An Introduction to the Interacting Boson Model of the Atomic Nucleus

Walter Pfeifer

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Dr. Walter Pfeifer : Street address: Stapfenackerweg 9
CH 5034 Suhr
Switzerland

email: wpfeifer@swissonline.ch

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Contents

Preface

The interacting boson model (IBM) is suitable for describing intermediate and heavy atomic nuclei. Adjusting a small number of parameters, it reproduces the majority of the low-lying states of such nuclei. Figure 0.1 gives a survey of nuclei which have been handled with the model variant IBM2. Figures 10.7 and 14.3 show the nuclei for which IBM1-calculations have been performed.

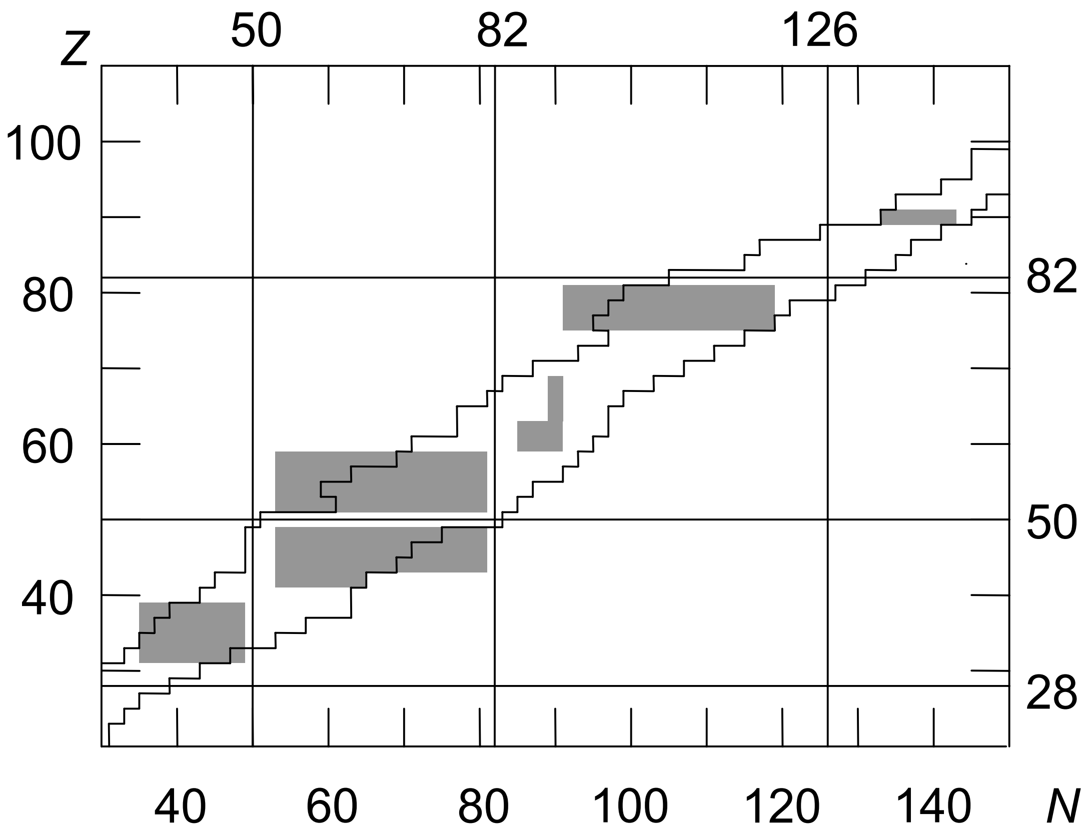

Figure 0.1. Card of even-even nuclei. Z = number of protons, N = number of neutrons. The dark areas denote nuclei which have been calculated using the IBM2 approximation (Iachello, 1988, p. 110).

The IBM is based on the well-known shell model and on geometrical collective models of the atomic nucleus. Despite its relatively simple structure, it has proved to be a powerful tool. In addition, it is of considerable theoretical interest since it shows the dynamical symmetries of several nuclei, which are made visible using Lie algebras.

The IBM was created in 1974 by F. Iachello and A. Arima (Arima and Iachello, 1975). Subsequently in numerous papers it has been checked, extended, and discussed. In 1990 in Santa Fe, New Mexico, Akito Arima was awarded the Weatherhill medal by the Franklin Institute for his many contributions to the field of nuclear physics. In the same year, Francesco Iachello received the Wigner medal given by the Group Theory and Fundamental Physics Foundation, which cited him for " developing powerful algebraic tools and models in nuclear physics ". In 1993, A. Arima and F. Iachello were awarded the T. W. Bonner Prize in Nuclear Physics by the American Physical Society.

The international symposia of Erice (Italy, 1978), Granada (Spain, 1981), Drexel (USA, 1983), Gull Lake (USA, 1984), La Rábida (Spain, 1985), Dubrovnik (Yugoslavia, 1986) as well as other events focusing on the IBM have clearly demonstrated the wide interest in this theory and its further development on an international scale. In 1994 in Padua (Italy), the "International Conference on Perspectives for the Interacting Boson Model on the Occasion of its 20th Anniversary" took place.

In recent years outstanding survey reports on the IBM have been published by Iachello and Arima (1987), Eisenberg and Greiner (1987), Talmi (1993), Frank and Van Isacker (1994) , and others. Unfortunately, there are few introductory books on the IBM available for the interested reader. The present publication might reduce this deficiency. It is directed towards physics students and experimental physicists interested in the main properties of the IBM. Knowledge of the elements of quantum mechanics, nuclear physics, and electrodynamics is a prerequisite.

The experienced reader may feel that some transformations and proofs have been rendered in too great detail. For the beginner, however, this is indispensable and may serve him as an exercise.

1 Introduction

A model of the atomic nucleus has to be able to describe nuclear properties such as spins and energies of the lowest levels, decay probabilities for the emission of gamma quantas, probabilities (spectroscopic factors) of transfer reactions, multipole moments and so forth. In this chapter those models are outlined from which the IBM comes.

The IBM is mainly rooted in the shell model, which has proved to be an excellent instrument for light nuclei (up to 50 nucleons). The larger the number of nucleons becomes the more shells have to be taken into account and the number of nuclear states soon becomes so colossal that the shell model will be intractable. For example the 2^+ state (spin 2 and positive parity) of ^{154}Sm shows $3{\cdot}10^{14}$ different configurations (Casten, 1990, p. 198). The interacting boson model (sometimes named interacting boson approximation IBA) reduces the number of states heavily. It constitutes only 26 configurations for the 2^+ state mentioned above.

The shell model reveals that the low-lying states of the even-even nuclei are made up predominantly by nucleon pairs with total spin 0 or 2. Higher spins of such pairs are rare for energy reasons (Hess, 1983, p. 55). Particularly the spins of pairs of identical nucleons are even numbers because they constitute an antisymmetric state (appendix A2). Furthermore, in the case of two identical nucleon pairs the total spin is strictly even, which follows from the fact that the pairs behave like bosons (see appendix A2). This theoretical result is not far from the real situation of even-even nuclei, from which it is known that their total spin predominantly is even.

These and other arguments led to the basic assumption of the IBM which postulates that the nucleon pairs are represented by bosons with angular momenta l = 0 or 2. The multitude of shells which appears in the shell model is reduced to the simple *s*-shell (l = 0) and the *d*-shell (l = 2) which is composed vectorially by *d*-bosons analogously to the shell model technique. The IBM builds on a closed shell i.e. the number of bosons depends on the number of active nucleon (or hole) pairs outside a closed shell. Each type of bosons, the *s*- and the *d*-boson, has its own binding energy with regard to the closed shell. Analogously to the standard shell model, the interacting potential of the bosons acts only in pairs.

As a peculiarity of the IBM there exist special cases in which certain linear combinations of matrix elements of this interaction potential vanish (chapters 10 and 14). In these cases the energies of the nuclear states and the configurations can be expressed in a closed algebraic form. These special cases are named "dynamic symmetries". They correspond to the well-known "limits" allocated to the vibration, the rotation et cetera of the whole nucleus. However most nuclei have to be calculated by diagonalising the Hamilton matrix as is usual in quantum mechanics (chapter 12).

The IBM is not only in connection with the shell model but also with the collective model of the atomic nucleus of Bohr and Mottelson (1953, 1975). In this model the deformation of the nuclear surface is represented by five parameters from which a Hamiltonian of a five dimensional oscillator results. It contains fivefold generating and annihilating operators for oscillator quanta. The operators of these bosons correspond to the operators of the *d*-shell in the IBM.

However, the handling of the collective model is laborious (Jolos, 1985, p. 121). Moreover, the number of bosons is unlimited and is not a good quantum number in contrast to the situation in the IBM. The special cases mentioned above are reproduced by some versions of geometric models but they are not joined together continuously (Barrett, 1981, p. 534). In the IBM these relations exist.

An additional relationship between both models consists in the fact that the form of the Hamilton operator (after suitable transformations) is similar to the one of the IBM (Jolos, 1985, p. 124).

2 Characteristics of the IBM

The simplest versions of the IBM describe the even-even nucleus as an inert core combined with bosons which represent pairs of identical nucleons. Bosons behave symmetrically in the following way: supposing that each boson has a wave function, that can be attributed, the wave function of the total configuration does not alter if two bosons (i.e. their variables) are interchanged. The analogy between nucleon pairs and bosons does not go so far that in the IBM the wave functions of the corresponding nucleons would appear. However, in the interacting boson-fermion model (chapter 16) which deals with odd numbers of identical nucleons, bosons are coupled to nucleons. Bosons are taken as states without detailed structure and their symmetry properties result in commutation relations for the corresponding creation- and annihilation operators (chapter 5).

The total spin of a boson is identical with its angular momentum i.e. one does not attribute an intrinsic spin to the bosons. Since the angular momenta of the bosons are even (l = 0, 2) their parity is positive. Although plausible arguments exist for these angular momenta mentioned in the foregoing chapter, this choice is arbitrary and constitutes a typical characteristic of the theory (however, exotic variants have been developed with l = 4 or odd values). Only the success achieved by describing real nuclei justifies the assumption for the angular momenta.

The models IBM1 and IBM2 are restricted to nuclei with even numbers of protons and neutrons. In order to fix the number of bosons one takes into account that both types of nucleons constitute closed shells with particle numbers ..28, 50, 82 and 126 (magic numbers). Provided that the protons fill less than half of the furthest shell the number of the corresponding active protons has to be divided by two in order to obtain the boson number N_π attributed to protons. If more than half of the shell is occupied the boson number reads N_π = (number of holes for protons)/2. By treating the neutrons in an analogous way, one obtains their number of bosons N_ν . In the IBM1 the boson number N is calculated by adding the partial numbers i.e. $N = N_\pi + N_\nu$. For example the nucleus $^{118}_{54}Xe_{64}$ shows the numbers N_π = (54 - 50)/2 = 2, N_ν = (64 -50)/2 = 7 and for $^{128}_{54}Xe_{74}$ the values N_π = (54 - 50)/2 = 2, N_ν = (82 - 74)/2 = 4 hold. Electromagnetic transitions don't alter the boson number but transfers of two identical nucleons lift or lower it by one.

Naturally the IBM has to take into account the fact that every nuclear state has a definite total nuclear angular momentum J or rather that the eigenvalue of the angular momentum operator $\boldsymbol{J}^2$ is $J(J + 1)\cdot\hbar$. J is an integer.

A boson interacts with the inert core of the nucleus (having closed shells) from which results its single boson energy ε. Three-boson interactions are excluded in analogy with the assumptions of the standard shell model. In contrast to the collective model, in the IBM one does not obtain a semiclassical, vivid picture of the nucleus but one describes the algebraic structure of the Hamiltonian operator and of the states, for which reason it is named an algebraic model.

3 Many-body configurations

At the beginning of this chapter the representation of boson configurations will be outlined and in the second section completely symmetric states of a few *d*-bosons will be formulated explicitly. In the end the rules are put together which hold for the collective states in the seniority scheme. They are compared with the results of section 3.2. In this chapter vector coupling technique is being applied, which is reviewed in the appendices A1 up to A3.

3.1 Many-boson states

Here we introduce a formulation of completely symmetric states of N bosons of which n_d have a *d*-state and N - n_d bosons are in the *s*-state. Besides the total angular momentum J and its projection M, for the most part additional ordering numbers are required in order to describe the collective state. One of these numbers is the seniority τ, after which the most usual representation scheme is named.

For the moment we are leaving out the additional ordering number and write the completely symmetric configuration symbolically as

$$| \, (s^{N-n_d})^{(0)}{}_0 \, (d^{\,n_d})^{(J)}{}_M, \, J\, M \, \rangle. \tag{3.1}$$

The *s*-bosons are coupled to a J = 0-state. In detail, the *d*-boson part is composed of single *d*-boson states having the angular momentum components 2, 1, 0, -1 and -2. These five single boson states d_μ appear in linear combinations as will be shown in the next section. The expression (3.1) is normalised to one.

3.2 Symmetric states of two and three *d*-bosons

In this section the *s*-bosons are left out of consideration and we will deal with the symmetrisation of configurations with a small number n_d of *d*-bosons.

First we take n_d = 2. According to the relation (A2.7) the configuration $| \, d^{\,2}, J\, M \, \rangle$ is symmetrical by itself if J is an even number. It has the form (A1.1)

$$| \, d^{\,2}, J\, M \, \rangle = | \, \textstyle\sum_{\mu_1\, \mu_2} (2\; \mu_1\; 2\; \mu_2 \,|\, J\, M\,)\; d_{\mu_1}\, d_{\mu_2} \, \rangle \equiv | \, [\, d \times d\,]^{(J)}{}_M \, \rangle, \quad J = 0, 2, 4. \tag{3.2}$$

In order to obtain a three-boson state we couple one *d*-boson to a boson pair which has an even angular momentum J_0 i.e. we form

$$| \, d^{\,3}, J_0\, J\, M \, \rangle \;\equiv | \, [[\, d \times d\,]^{(J_0)} \times d\,]^{(J)}{}_M \, \rangle. \tag{3.3}$$

This expression is considered as a fully symmetrical three-*d*-boson state, which is obtained by carrying out a transposition procedure. In order to formulate this method, temporarily we are regarding bosons as distinguishable and we attribute an individual number to each single boson state. Supposing that such a state is described by a wave function, we have to label every variable with this boson number. We make use of the relation $[\, d(1) \times d(2)]^{(J_0)} = [\, d(2) \times d(1)\,]^{(J_0)}$, which holds for an even J_0 according to (A2.5) and (A2.7). Starting from the partially symmetric (p. s.) form

$$|\,[\, d(1) \times d(2)\,]^{(J_0)} \times d(3)\,]^{(J)}{}_M \rangle_{\text{p.s.}},$$

we obtain a symmetric three-boson state by adding two analogous forms in which the last d-boson is substituted as follows

$$A^{-1}\,|\,[[\, d \times d\,]^{(J_0)} \times d\,]^{(J)}{}_M \rangle = |\,[[\, d(1) \times d(2)\,]^{(J_0)} \times d(3)\,]^{(J)}{}_M \rangle_{\text{p.s.}} +$$

$$|\,[[\, d(1) \times d(3)\,]^{(J_0)} \times d(2)\,]^{(J)}{}_M \rangle_{\text{p.s.}} + |\,[[\, d(3) \times d(2)\,]^{(J_0)} \times d(1)\,]^{(J)}{}_M \rangle_{\text{p.s.}}\,. \qquad (3.4)$$

A is the normalisation factor of the right hand side of (3.4). This expression is symmetric because one reproduces it by interchanging two boson numbers (for example 2 and 3). In the last but one term, we can interchange $d(1)$ and $d(3)$ because it is partially symmetric. We employ the recoupling procedure (A3.3) and (A3.6) to the last two terms in (3.4) and obtain

$$A^{-1}\,|\,[[\, d \times d\,]^{(J_0)} \times d\,]^{(J)}{}_M \rangle = |\,[[\, d(1) \times d(2)\,]^{(J_0)} \times d(3)\,]^{(J)}{}_M \rangle_{\text{p.s.}} + \qquad (3.5)$$

$$\textstyle\sum_{J'} (-1)^J\cdot\sqrt{(2J_0 + 1)}\cdot\sqrt{(2J' + 1)}\cdot\{{}^2{}_2\;{}^2{}_J\;{}^{J_0}{}_{J'}\}\cdot|\,[\, d(3) \times [\, d(1) \times d(2)\,]^{(J')}]^{(J)}{}_M \rangle_{\text{p.s.}} +$$

$$\textstyle\sum_{J'} (-1)^J\cdot\sqrt{(2J_0 + 1)}\cdot\sqrt{(2J' + 1)}\cdot\{{}^2{}_2\;{}^2{}_J\;{}^{J_0}{}_{J'}\}\cdot|\,[\, d(3) \times [\, d(2) \times d(1)\,]^{(J')}]^{(J)}{}_M \rangle_{\text{p.s.}}.$$

We now interchange $d(2)$ and $d(1)$ in the last term of (3.5), which yields the factor $(-1)^{J'}$ ((A1.4)). Both sums are added then, through which all terms with odd values J' disappear. In the resulting sum we interchange $d(3)$ and $[\, d(2) \times d(1)\,]^{(J')}$, which annihilates the factor $(-1)^J$ according to (A1.4), because J' is even. For formal reasons the first term on the right hand side of (3.5) is replaced by $\sum_{J'\text{even}} \delta_{J'J_0}\,|\,[[\, d(1) \times d(2)\,]^{(J')} \times d(3)\,]^{(J)}{}_M \rangle_{\text{p.s.}}$. One obtains

$$A^{-1}\,|\,[[\, d \times d\,]^{(J_0)} \times d\,]^{(J)}{}_M \rangle = \qquad (3.6)$$

$$\textstyle\sum_{J'\text{even}} (\delta_{J'J_0} + 2\cdot\sqrt{(2J_0 + 1)}\cdot\sqrt{(2J' + 1)}\cdot\{{}^2{}_2\;{}^2{}_J\;{}^{J_0}{}_{J'}\})\,|\,[[\, d(1) \times d(2)\,]^{(J')} \times d(3)\,]^{(J)}{}_M \rangle.$$

The normalisation factor A reads

$$A = (3 + 6(2J_0 + 1)\,\{{}^2{}_2\;{}^2{}_J\;{}^{J_0}{}_{J_0}\})^{-1/2}. \qquad (3.7)$$

It's a good exercise to derive this expression explicitly. The state $|\,[[\, d \times d\,]^{(J_0)} \times d\,]^{(J)}{}_M \rangle$ is regarded as normalised to one. Analogously to the two-boson states (A1.9) here the partially symmetric states $|[[d(1)\times d(2)]^{(J')}\times d(3)]^{(J)}{}_M\rangle$ with different J' are orthogonal to each other. We employ a slightly modified form of (3.5)

$$A^{-1}\,|\,[[\, d \times d\,]^{(J_0)} \times d\,]^{(J)}{}_M \rangle = |\,[[\, d(1) \times d(2)\,]^{(J_0)} \times d(3)\,]^{(J)}{}_M \rangle_{\text{p.s.}} +$$

$$\textstyle\sqrt{(2J_0 + 1)}\cdot\sum_{J'} (1 + (-1)^{J'})\sqrt{(2J' + 1)}\cdot\{{}^2{}_2\;{}^2{}_J\;{}^{J_0}{}_{J'}\}\cdot|\,[[\, d(1) \times d(2)\,]^{(J')} \times d(3)\,]^{(J)}{}_M \rangle_{\text{p.s.}}\,.$$

and make up the following equation

$$A^{-2} \langle [[d \times d]^{(J_0)} \times \mathrm{d}]^{(J)}{}_{\mathrm{M}} \mid [[\, d \times d\,]^{(J_0)} \times d\,]^{(J)}{}_{M} \rangle = A^{-2} =$$

$$1 + 2{\cdot}2{\cdot}(2J_0 + 1)\begin{Bmatrix} 2 & 2 & J_0 \\ 2 & J & J_0 \end{Bmatrix} + (2J_0 + 1) \textstyle\sum_{J'} (2 + 2{\cdot}(-1)^{J'})(2J' + 1)\begin{Bmatrix} 2 & 2 & J_0 \\ 2 & J & J' \end{Bmatrix}^2.$$

Due to (A3.8) and (A3.9) the relations

$$\textstyle\sum_{J'} (2J' + 1) \begin{Bmatrix} 2 & 2 & J_0 \\ 2 & J & J' \end{Bmatrix}^2 = (2J_0 + 1)^{-1} \text{ and}$$

$$\textstyle\sum_{J'} (-1)^{J'}(2J' + 1)\begin{Bmatrix} 2 & 2 & J_0 \\ 2 & J & J' \end{Bmatrix}^2 = \begin{Bmatrix} 2 & 2 & J_0 \\ 2 & J & J_0 \end{Bmatrix} \text{ hold, from which we derive}$$

$$A^{-2} = 1 + 2 + (2 + 4)(2J_0 + 1)\begin{Bmatrix} 2 & 2 & J_0 \\ 2 & J & J_0 \end{Bmatrix}, \text{ which is in agreement with (3.7).}$$

We now look into the J-values of symmetric three d-boson states represented in (3.6).

The case $J = 0$ is of some importance in the seniority scheme. The number of triplets with $J = 0$ is named n_Δ i.e. in this case we have $n_\Delta = 1$.

For $J \neq 0$ we insert the numerical values of the 6-j symbols (A3.12 - 14) in the equation (3.6). For $J = 1$ the partial vectors can only show $J_0 = J' = 2$ and the expression (3.6) vanishes. For $J = 2$ the values $J_0 = 0, 2, 4$ have to be considered and the calculation yields

$$\mid [[\, d \times d\,]^{(0)} \times d\,]^{(2)}{}_M \rangle = \mid [[\, d \times d\,]^{(2)} \times d\,]^{(2)}{}_M \rangle = \mid [[\, d \times d\,]^{(4)} \times d\,]^{(2)}{}_M \rangle\ .$$

We take a special interest in states with $J_0 = 0$, that is why we treat the state ($J_0 = 2$, $J = 2$) mentioned above as equivalent to ($J_0 = 4$, $J = 2$) and to ($J_0 = 0$, $J = 2$) . Therefore we say, the configuration ($J_0 \neq 0$, $J = 2$) does not exist. In a similar way we see that the $J = 3$-states ($J_0 = 2, 4$) differ only in their signs. Both $J = 4$-states ($J_0 = 2, 4$) are identical. For $J = 5$ ($J_0 = 4$) the expression (3.6) vanishes. $J = 6$ characterises the so-called "stretched" state.

3.3 The seniority scheme, rules defining *J*

General symmetric states of d-bosons are constructed by vector coupling and complete symmetrisation using group theory (Hamermesh, 1962), (Bayman and Landé, 1966). Here we have a look at the seniority scheme, which is the most common version of this representation. The configuration of n_d d-bosons is written as follows

$$\mid n_d\, , ([d \times d]^{(0)}{}_0)^{n_\pi}{\cdot}([[d \times d]^{(2)} \times d]^{(0)}{}_0)^{n_\Delta} \cdot (d^\lambda)^{(J)}{}_M \rangle\ . \qquad (3.8)$$

In the expression (3.8) the doublet $[d \times d]^{(0)}{}_0$ with angular momentum 0 appears n_π times and the triplet $[[d \times d]^{(2)} \times d]^{(0)}{}_0$ exists n_Δ times. The λ remaining d-bosons constitute a configuration with the total angular momentum J (M) which contains neither a doublet nor a triplet with $J = 0$. Therefore the number of d-bosons reads $n_d = 2n_\pi + 3n_\Delta + \lambda$. The number $\tau = n_d - 2n_\pi = 3n_\Delta + \lambda$, which is left over after subtracting the doublets, is named seniority analogously to the description in the shell model. We name the configuration $(d^\lambda)^{(J)}{}_M$ the "reduced" state of the λ bosons. It is defined unambiguously by λ, J and M (Talmi, 1993,

S. 763). Its total angular momentum J is identical with the one of the whole configuration (3.8).

In the seniority scheme the *d*-boson configurations are defined by the numbers n_d, n_π, n_Δ, J, (M).

There exist restrictions for the J-values. It can be shown that in a "reduced" state of λ d-bosons the following values are permitted

$$J = \lambda,\ \lambda + 1,\ \ldots\ ,\ 2\lambda - 3,\ 2\lambda - 2,\ 2\lambda, \tag{3.9}$$

i. e. $J < \lambda$ and $J = 2\lambda - 1$ are inadmissible. $J = 2\lambda$ represents the "stretched" state.

The exclusion of $J = 2\lambda - 1$ in (3.9) can be explained in the following way. We know that for the "stretched", symmetric and to z orientated state of λ d-bosons the relation $J = M = 2\lambda$ holds. We now construct the symmetric state with $M = 2\lambda - 1$ and represent it using numbered bosons whose projections of the angular momentum is μ :

$$\begin{aligned} |(d^{\lambda})_{M = 2\lambda - 1}\rangle = A\ |(& d(1)_{\mu=1}\cdot d(2)_{\mu=2}\cdot d(3)_{\mu=2}\cdot\ \ldots\ \cdot d(\lambda)_{\mu=2} \\ &+ d(1)_{\mu=2}\cdot d(2)_{\mu=1}\cdot d(3)_{\mu=2}\cdot\ \ldots\ \cdot d(\lambda)_{\mu=2} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ d(1)_{\mu=2}\cdot d(2)_{\mu=2}\cdot d(3)_{\mu=2}\cdot\ \ldots\ \cdot d(\lambda)_{\mu=1})\rangle\ . \end{aligned} \tag{3.10}$$

A is the normalisation constant. The expression (3.10) reveals that there exists only one state with $M = 2\lambda - 1$. On the other hand, if one turns the "stretched" state (with $J = 2\lambda$) relative to the z-axis in order to obtain the projection $M = 2\lambda - 1$, the resulting state is still symmetric and must agree with the one of (3.10) because this is unique. For the same reason a state with $J = 2\lambda - 1$ is not allowed because its maximal projection would be $M = 2\lambda - 1$ which must not occur twice.

We now verify the rule (3.9) inspecting the boson states (3.2) and (3.6). For $\lambda = 2$ the "reduced" state reads $|\,[\,d \times d\,]^{(J)}{}_M\,\rangle$ with $J \neq 0$. Owing to (3.9) only the values $J = 2, 4$ have to be considered which is in agreement with (3.2). For the "reduced" state with $\lambda = 3$ according to (3.9) the values $J = 0, 1, 2$ are ruled out. In fact the discussion of equation (3.6) showed that $J = 1$ does not appear and that both other cases are equivalent to $J_0 = 0$ which is inconsistent with the term "reduced" state. The rule (3.9) excludes $J = 5$ which has been found to be true for $\lambda = 3$. Thus, for $\lambda = 2$ and 3 the selection rule (3.9) is confirmed.

In table 3.1 for several boson numbers n_d the allowed values n_π and n_Δ are given. Accompanying values for τ, λ and J are in the columns 3, 5 and 6.

Table 3.1. Classification of the d-boson configurations in the seniority scheme.
n_d : number of d-bosons,
n_π : number of boson pairs with total angular momentum 0,
n_Δ : number of boson triplets with total angular momentum 0,
τ : seniority, λ : number of bosons in the "reduced" state,
J : total angular momentum

n_d	n_π	$\tau = n_d - 2n_\pi$	n_Δ	$\lambda = \tau - 3n_\Delta$	J
2	0	2	0	2	2, 4
2	1	0	0	0	0
3	0	3	0	3	3, 4, 6
3	0	3	1	0	0
3	1	1	0	1	2
4	0	4	0	4	4, 5, 6, 8
4	0	4	1	1	2
4	1	2	0	2	2, 4
4	2	0	0	0	0
.	.	.	.	.	.
.	.	.	.	.	.
7	0	7	0	7	7,8,9,10,11,12,14
7	0	7	1	4	4, 5, 6, 8
7	0	7	2	1	2
7	1	5	0	5	5,6,7,8,10
7	1	5	1	2	2, 4
7	2	3	0	3	3, 4, 6
7	2	3	1	0	0
7	3	1	0	1	2
.	.	.	.	.	.

Table 3.1 shows that for given $n_d > 3$ some angular momenta J appear in more than one configuration. The value $J = 1$ is absent in the whole spectrum. Clearly it is missing also for $\lambda = 1$ because this simplest "reduced" state consists of a single d-boson.

Among states with several d-bosons it happens that configurations with equal (n_d, τ, J)-values differ in the quantity n_Δ and are not orthogonal to one another. They have to be orthogonalised with the help of the well-known Schmidt procedure. By doing it, the number n_Δ looses its character of an ordering number and it has to be replaced by an arbitrarily defined index.

Many-boson configurations in the seniority scheme stand out because they are eigenfunctions of the vibrational limit of the Hamilton operator (chapter 10 and section 14.4). Since this special case correlates with the Lie algebra $u(5)$ the states of the seniority scheme in addition are named $u(5)$-basis. "Spherical basis " is a further customary name. Besides this scheme there exist two less often used representations which are eigenfunctions of other limits of the Hamiltonian (chapter 14).

4 Many-boson states with undefined angular momentum

In this chapter we will deal with the simplest representation of many-boson states. It is formed as a product of single-boson state functions. Symmetry is achieved by permuting the boson indices and adding up the resulting expressions. In contrast to (3.8), the single angular momenta are not coupled to a definit total quantity. We nam this representation "primitive" basis.

In the first section we will look into such representations of a few bosons and relate them to the formulations with defined angular momentum. The symmetric primitive representation of a multitude of different single-boson states is treated in the second section.

4.1 Two- and three-*d*-boson states

By way of introduction, we treat the "primitive" states of a few *d*-bosons. The single *d*-boson state is characterized by $d_m(n)$. The quantity m is the projection of the angular momentum l = 2 and n is the number of the bosons. Provisionally we are considering the bosons as distinguishable.

The symmetric and normalised two-*d*-boson state of this kind is represented by $| d\, m\, d\, m' \rangle$, in which single boson states such as $d_m(i)$ and $d_{m'}(k)$ are involved. It has the form

$$| d\, m\, d\, m' \rangle = (2(1 + \delta_{mm'}))^{-1/2}\, | d_m(1)\, d_{m'}(2) + d_m(2)\, d_{m'}(1) \rangle. \tag{4.1}$$

Interchanging 1 and 2 has no influence and $\langle d\, m\, d\, m' | d\, m\, d\, m' \rangle = 1$ holds i.e. the expression (4.1) is symmetric and normalised.

Correspondingly, "primitive" three-*d*-boson states are constructed by summing up $| d_m(1)\, d_{m'}(2)\, d_{m''}(3) \rangle$, $| d_m(1)\, d_{m'}(3)\, d_{m''}(2) \rangle$ and $| d_m(2)\, d_{m'}(1)\, d_{m''}(3) \rangle$ etc and by normalising. One writes

$$| \mathrm{d}\, m\, d\, m'\, d\, m'' \rangle = A_p(m, m', m'') \cdot \Sigma_{\text{all permutations}} | d_m(1)\, d_{m'}(2)\, d_{m''}(3) \rangle. \tag{4.2}$$

The permutations concern the numbers 1, 2 and 3. The normalisation factor $A_p(m, m', m'')$ is $1/\sqrt{6}$ if all quantities m are different. Supposing that only two of them agree, $A_p = 1/(2\sqrt{3})$ holds and for three identical single boson states we have $A_p = 1/6$.

Before we turn to the general representation of the "primitive" basis we show the connection between the states (4.1-2) and those of the seniority scheme (3.2), (3.3) and (3.8) whose angular momenta are good quantum numbers.

According to equation (3.2) the symmetric two-boson state of this kind reads

$$| d^2\, J\, M \rangle \equiv |[d \times d]^{(J)}{}_M \rangle = \Sigma_{m\, m'} (2\, m\, 2\, m' | J\, M)\, | d_m(1)\, d_{m'}(2) \rangle. \tag{4.3}$$

The angular momentum J is even. We introduce an operator $\boldsymbol{S}$ which interchanges in equation (4.3) the indices 1 and 2 and adds the new expression to the original one. Due to the symmetry in (4.3) the following relations hold

$$| d^2 J M \rangle = \tfrac{1}{2}\cdot\boldsymbol{S}\, |d^2 J M \rangle =$$

$$\tfrac{1}{2}\cdot \textstyle\sum_{m\neq m'} (2\; m\; 2\; m' \,|J\; M)\; (\sqrt{2}/\sqrt{2})\cdot|d_m(1)\; d_{m'}(2) + d_m(2)\; d_{m'}(1)\rangle\; +$$

$$\tfrac{1}{2}\cdot(2\; M/2\; 2\; M/2\; |J\; M)\cdot 2\cdot|d_{M/2}(1)\; d_{M/2}\,(2)\rangle\cdot\delta_{M\,\mathrm{even}} =$$

$$\textstyle\sum_{m\neq m'} (2\; m\; 2\; m' \,|J\; M)(1/\sqrt{2})\cdot|d\; m\; d\; m'\rangle\; +$$

$$(2\; M/2\; 2\; M/2\; |J\; M)\cdot|d\; M/2\; d\; M/2\rangle\cdot\delta_{M\,\mathrm{even}} =$$

$$\textstyle\sum_{mm'} \sqrt{((1 + \delta_{mm'})/2)}\; (2\; m\; 2\; m' \,|\; J\; M)\cdot|d\; m\; d\; m' \rangle \qquad (4.4)$$

The last expression in (4.4) is built with "primitive" states introduced in (4.1).

The symmetric states of three-d-bosons with defined angular momentum J can be treated in a similar way. According to (3.4) we have

$$A^{-1}\, | d^3 J_0\, J\, M \rangle = | [[d(1) \times d(2)]^{(J_0)} \times d(3)]^{(J)}{}_M \rangle\; +$$

$$| [[d(2) \times d(3)]^{(J_0)} \times d(1)]^{(J)}{}_M \rangle\; +$$

$$| [[d(3) \times d(1)]^{(J_0)} \times d(2)]^{(J)}{}_M \rangle. \qquad (4.5)$$

The quantity J_0 is even and the normalisation factor A is given in (3.7). Using the Clebsch-Gordan coefficients defined in (A1.1) we rewrite (4.5) and in the following step we insert the "primitive" representation of (4.2) . We obtain

$$A^{-1}\, | d^3 J_0\, J\, M \rangle = \textstyle\sum_{M_0 m''} (J_0\; M_0\; 2\; m'' \,|\; J\; M)\cdot \qquad (4.6)$$

$$\textstyle\sum_{mm'} (2\; m\; 2\; m' \,|\; J_0\; M_0)\cdot\tfrac{1}{2} \sum_{\text{all permutations}} |\; d_m(1)\; d_{m'}(2)\; d_{m''}(3) \rangle =$$

$$\textstyle\sum_{M_0 m''} (J_0\; M_0\; 2\; m'' \,|J\; M)\cdot\sum_{mm'} (2\; m\; 2\; m' \,|\; J_0\; M_0)\cdot\tfrac{1}{2}\cdot A_p(m,m',m'')^{-1}\cdot|\; d\; m\; d\; m'\; d\; m'' \rangle.$$

In (4.4) and (4.6) the states of the seniority scheme are written as a linear combination of "primitive" states. It's obvious that this is true for all many-boson states. This stimulates further research on the "primitive" basis. In doing so, we will encounter the well-known creation- and annihilation operators for bosons (chapter 5), which facilitate an elegant representation of the boson interactions.

4.2 General "primitive" many-boson states

In the interacting boson model the bosons occupy the following six single states s, d_2, d_1, d_0, d_{-1} and d_{-2} . For the sake of simplicity, we have a look at an arbitrary number of single boson states, which we represent by ψ_a, ψ_b, .. , ψ_f, .. . In essence, we follow the report of Landau (86).

A single boson state can be occupied by several bosons. The number of bosons which are in the single state ψ_f is named N_f . The total number of bosons N amounts to the sum of these partial numbers

$$N = N_a + N_b + ... + N_f + .. \quad . \tag{4.7}$$

The ansatz for the state Ψ of N bosons is written as a product of N single boson states which contains N_a times the factor ψ_a, N_b times the factor ψ_b, etc. Each single state is designed by a different boson number as follows

$$\Psi(N_a, N_b, .. ,N_f, .) = \Pi_{i=1}{}^{N_a}\psi_a(i) \cdot \Pi_{i=Na+1}{}^{N_a+N_b}\psi_b(i) \cdot ... \cdot \Pi_{i=Ne+1}{}^{N_a+..+N_f}\psi_f(i) \cdot ... \quad . \tag{4.8}$$

The index i in the partial products marks the bosons occupying the affiliated single state. The function Ψ does not alter if the labels of bosons are interchanged which belong to the same single state. In order to obtain complete symmetry we interchange bosons in all possible ways, which belong to different single states. All these configurations are generated by arranging the N numbered bosons in the single states $a, b, ..,f,..$ with the partial numbers N_a, N_b, .. ,N_f ..in all different ways. In other words, we have here the combinatorial task to calculate how often N different balls can be put in vessels dissimilarly in such a way that the first one contains N_a balls, the second one N_b balls, etc. From the combinatorial analysis we take that the number of possibilities or permutations reads

$$N!/(N_a!\cdot N_b!\cdot .. N_f!\cdot ..). \tag{4.9}$$

We label these permutations by the index r and introduce the operator $\boldsymbol{P'_r}$ which performs the rth permutation when acting on the state $\Psi(N_a, N_b,.., N_f, .)$. We will show that the state

$$A^{-1} \mid N_a, N_b, ... ,N_f, .. \rangle = \sum_r \boldsymbol{P'_r}\Psi(N_a, N_b,.., N_f,..) \tag{4.10}$$

is completely symmetric with regard to interchanging numbered bosons. We can see this by interchanging two bosons which belong to different single states. Since every summand in (4.10) has a counterpart which contains the mentioned bosons in the interchanged positions, the transposition has no effect on the whole sum. This is true for all kinds of exchange.

We now turn to the normalisation constant A in (4.10). We assume that the single states $\psi_f(i)$ depend on variables which we represent by the symbol ξ_i . We regard these states as orthonormalised i. e.

$$\int \psi_g{}^*(\xi_i)\psi_f(\xi_i)\, d\xi_i = \delta_{gf} \quad . \tag{4.11}$$

We maintain that all functions $\boldsymbol{P'_r}\Psi(N_a, N_b,.., N_f, .)$ are orthonormal with respect to each other as follows

$$\int [\, \boldsymbol{P'_r}\Psi(N_a, N_b,.., N_g,..)]^*\, \boldsymbol{P'_{r'}}\,\Psi(N_a, N_b,.., N_f, .)d\xi_1 .. d\xi_N = \delta_{\boldsymbol{r}\boldsymbol{r'}}. \tag{4.12}$$

At least one boson (say the ith) can be found, namely, which is assigned to different single states (for instance the states f and g) in the functions

$\boldsymbol{P'_r}\Psi(N_a, N_b,.., N_g,..)$ and $\boldsymbol{P'_{r'}}\,\Psi(N_a, N_b,.., N_f, .)$. As a result, among other things the partial integral $\int\psi_g{}^*(\xi_i)\psi_f(\xi_i)d\xi_i$ $(g \neq f)$ vanishes. With that, the orthogonality of the functions $\boldsymbol{P'_r}\Psi(N_a, N_b,.., N_f,..)$ is shown. Taking into account the number of special permutations $\boldsymbol{P'_{r'}}$ (4.9) and using (4.12) we obtain

$$\langle\, N_a, N_b,.., N_f, . \mid N_a, N_b,.., N_f,.. \,\rangle = A^2 \cdot N!/(N_a! \cdot N_b! \cdot\, ..\, N_f! \cdot\, ..). \qquad (4.13)$$

The left-hand side of (4.13) has to be 1 i. e.

$$A = (N_a! \cdot N_b! \cdot\, ..\, N_f! \cdot\, ../N!)^{\frac{1}{2}} \qquad (4.14)$$

and $$\mid N_a, N_b,.., N_f,.. \,\rangle = (N_a! \cdot N_b! \cdot\, ..\, N_f! \cdot\, ../N!)^{\frac{1}{2}} \cdot \sum_r \boldsymbol{P'_r}\Psi(N_a, N_b,.., N_f,..). \qquad (4.15)$$

This function describes the " primitive" state of N bosons with single states a, b, .. , f, .. containing the boson numbers $N_a, N_b,.., N_f$, .. . For N = 2 and 3 one can show directly the agreement between (4.15) and (4.1) or (4.2). Doing so, one has to take into account that in (4.2) **all** permutations appear.

5 Operators and matrix elements

The interacting boson model and other many-particle models deal with operators which act on single boson states or on pair states. Because in our model the representation of the collective state is symmetric for all N bosons the mentioned operators cannot be directed towards individual, labelled bosons. However, they act on the collective of bosons and correspondingly they have a symmetric form. The single-boson operator $\boldsymbol{F}^{(1)}$ reads

$$\boldsymbol{F}^{(1)} = \sum_{i=1}^{N} \boldsymbol{f}^{(1)}(\xi_i). \tag{5.1}$$

The operator $\boldsymbol{f}^{(1)}$ acts on a single boson which we consider to be distinguishable provisionally and to which we assign the index i. The argument ξ_i describes vectorial, local and other variables of the ith boson on which $\boldsymbol{f}^{(1)}(\xi_i)$ acts. For example, the operator of the kinetic energy or the operator of a fixed potential has the form of $\boldsymbol{F}^{(1)}$.

The two-boson operator reads

$$\boldsymbol{F}^{(2)} = \sum_{i>j=1}^{N} \boldsymbol{f}^{(2)}(\xi_i,\xi_j). \tag{5.2}$$

The partial operator $\boldsymbol{f}^{(2)}$ acts on the variables of a boson pair. For example, the potential operating in twos between bosons has the form (5.2). In simple models $\boldsymbol{f}^{(2)}$ is only a function of the difference $|\xi_i - \xi_j|$. The Hamilton operator, which is decisive for quantum mechanical problems, contains the operator types (5.1) and (5.2) in a linear combination.

In the following section we will deal with matrix elements of single boson operators using the representation of the "primitive" basis. Almost unavoidably, one reaches the definition of creation and annihilation operators (section 5.2). In the third section we will show how these operators contribute to the form of the matrix elements.

5.1 Matrix elements of the single-boson operator

In order to calculate eigenfunctions and eigenvalues of the Hamilton operator, matrix elements of the operators (5.1) and (5.2) have to be made up. Represented in terms of the "primitive" basis these matrix elements have the following form

$$\langle N_a, N_b, \ldots, N_f, \ldots | \boldsymbol{F} | N_a', N_b', \ldots, N_f', \ldots \rangle . \tag{5.3}$$

First, we deal with the single-boson operator $\boldsymbol{F}^{(1)}$. For the sake of simplicity we admit only two single states a and b. In the matrix element

$$\langle N_a, N_b | \boldsymbol{F}^{(1)} | N_a', N_b' \rangle \tag{5.4}$$

the condition $N_a + N_b = N_a' + N_b'$ holds because both total states belong to the same basis only if the total number of bosons is equal.

We begin with the **diagonal** matrix element, of which both functions are identical as follows

$$\sum_{i=1}^{N} \langle N_a, N_b \mid \boldsymbol{f}^{(1)}(\xi_i) \mid N_a, N_b \rangle. \tag{5.5}$$

First we take i = 1 and ask how many terms exist in the state $\mid N_a, N_b \rangle$ according to (4.15) and (4.9) in which the first boson is in the single state a. It is the number of permutations $\boldsymbol{P}_r'$ (see section 4.2) of the residual N – 1 bosons. Therefore there are $(N - 1)!/((N_a - 1)!N_b!)$ terms in the state function $\mid N_a, N_b \rangle$ with the first boson in the state a. In this case in (5.5) the states $\psi_a(\xi_1)$ and $\psi_a{}^*(\xi_1)$ constitute a separate integral together with $\boldsymbol{f}^{(1)}$. The integration over the residual factors containing the variables $\xi_2, \xi_3, .. \xi_N$ yields the value 1 (apart from the normalisation constant) if both factors agree in the allocation of bosons, otherwise it vanishes. Therefore the part of the diagonal matrix element in which the first boson is in the state a reads

$$\sqrt{(N_a!N_b!/N!)}\cdot\sqrt{(N_a!N_b!/N!)}\cdot\int \psi_a{}^*(\xi_1)\boldsymbol{f}^{(1)}(\xi_1)\psi_a(\xi_1)\, d\xi_1\, (N - 1)!/((N_a - 1)!N_b!) =$$

$$(N_a/N) \int \psi_a{}^*(\xi_1)\boldsymbol{f}^{(1)}(\xi_1)\psi_a(\xi_1)\, d\xi_1 \,. \tag{5.6}$$

The first factors in (5.6) are normalisation factors according to (4.14). Furthermore, in diagonal matrix elements the first boson must not appear in different states (a and b) together with $\boldsymbol{f}^{(1)}(\xi_1)$ in the partial integral, because then the residual states are orthogonal. On the other hand, one obtains an expression analogous to (5.6) if the first boson is twice in the state b . Consequently for i = 1 the diagonal matrix element reads

$$N^{-1}\cdot(N_a \cdot\int \psi_a{}^*(\xi_1)\boldsymbol{f}^{(1)}(\xi_1)\psi_a(\xi_1)\, d\xi_1 + N_b \cdot\int \psi_b{}^*(\xi_1)\boldsymbol{f}^{(1)}(\xi_1)\psi_b(\xi_1)\, d\xi_1). \tag{5.7}$$

Since according to (5.1) the operator $\boldsymbol{F}^{(1)}$ contains a sum including all i $(1 \le i \le N)$ and because the expression

$$\int \psi_f{}^*(\xi_i)\boldsymbol{f}^{(1)}(\xi_i)\psi_f(\xi_i)\, d\xi_i \equiv f_{ff}{}^{(1)} \tag{5.8}$$

does not depend on the boson number i, the expression (5.7) appears N times and the diagonal element reads

$$\langle N_a, N_b \mid \boldsymbol{F}^{(1)} \mid N_a, N_b \rangle = N_a f_{aa}{}^{(1)} + N_b f_{bb}{}^{(1)}. \tag{5.9}$$

If several single states a, b, c, .. , f, .. are involved, correspondingly we have

$$\langle N_a, N_b, .., N_f, ..\mid \boldsymbol{F}^{(1)} \mid N_a, N_b, .., N_f, ..\rangle = N_a f_{aa}{}^{(1)} + N_b f_{bb}{}^{(1)} + .. + N_f f_{ff}{}^{(1)}.. \tag{5.10}$$

We now deal with **non-diagonal** matrix elements of the single-boson operator $\boldsymbol{F}^{(1)}$. Again we restrict the single states to a and b. We bring the total states in the following sequence in which the occupation numbers N_a and N_b are stressed

occupation numbers		
state a	state b	
0	N	
1	N - 1	
.	.	
N_a - 1	N_b + 1	(5.11)
N_a	N_b	
N_a + 1	N_b - 1	
.	.	
N	0	

with $N = N_a + N_b$. It turns out that non-diagonal (belonging to two different lines in (5.11)) matrix elements are different from zero only if the occupation numbers of one single state on both sides of (5.4) differ in 1. In order to show this we treat

$$\sum_{i=1}^{N} \langle N_a, N_b \mid \boldsymbol{f}^{(1)} (\xi_i) \mid N_a + 1, N_b - 1 \rangle. \tag{5.12}$$

We begin also with $i = 1$ and determine the number of summands on the right hand side of (5.12) in which the first boson occupies the **state *a***. Again it results from the number of permutations $\boldsymbol{P_r}'$ of the residual N - 1 bosons i. e. there are $(N - 1)!/(N_a!(N_b - 1)!)$ terms of this kind. On the left hand side of (5.12) the number of terms showing the first boson in the **single state *b*** is $(N - 1)!/(N_a!(N_b - 1)!)$ as well. These are the only terms which we have to take into account if we form the integral in (5.12) in demanding $i = 1$. We separate the factors $\psi_b{}^*(\xi_1)$, $\psi_a(\xi_1)$ and $\boldsymbol{f}^{(1)}$ from the rest and carry out the integral over ξ_1. Analogously to (5.6) the integration of the residual function depending on ξ_2, ξ_3.. ξ_N yields the value 1 (apart from the normalisation constant) if both factors agree in the allocation of bosons which happens $(N - 1)!/(N_a!(N_b - 1)!)$ times. The integral vanishes if the residues disagree. Therefore the part of the matrix element in which the first boson is on the left hand side in the state b and on the right hand side in a reads as follows

$$\langle N_a, N_b \mid \boldsymbol{f}^{(1)} (\xi_1) \mid N_a +1, N_b -1 \rangle =$$

$$\surd(N_a!N_b!/N!)\cdot\surd((N_a +1)!(N_b -1)!/N!) \int\psi_b{}^*(\xi_1)\boldsymbol{f}^{(1)}(\xi_1)\psi_a(\xi_1)\, d\xi_1\cdot(N -1)!/((N_a!(N_b -1)!) =$$

$$\surd(N_a +1)\surd N_b/N\cdot\int \psi_b{}^*(\xi_1)\boldsymbol{f}^{(1)}(\xi_1)\psi_a(\xi_1)\, d\xi_1. \tag{5.13}$$

We take into account that $\boldsymbol{F}^{(1)}$ contains a sum over all N bosons and that the integral

$$\int \psi_b{}^*(\xi_1)\boldsymbol{f}^{(1)}(\xi_1)\psi_a(\xi_1)\, d\xi_1 \equiv f_{ba}{}^{(1)} \tag{5.14}$$

does not depend on the boson number i. Thus we obtain

$$\langle N_a, N_b \mid \boldsymbol{F}^{(1)} \mid N_a +1, N_b -1 \rangle = \surd(N_a +1)\surd N_b \cdot f_{ba}{}^{(1)} . \tag{5.15}$$

The matrix elements of a basis are arranged usually in a quadratic matrix in which the x-direction indicates the increasing order of the right hand side part of (5.4) and the falling y-direction is associated with the left hand side state. In this representation the elements given by (5.15) lie directly beside the diagonal which is why we name them off diagonal. The matrix elements lying directly on the other side of the diagonal are formulated analogously to the equation (5.15).

Matrix elements of the type $\langle N_a, N_b \mid \boldsymbol{F}^{(1)} \mid N_a + 2, N_b - 2 \rangle$ and those which are more distant from the diagonal all vanish because the residual functions mentioned above never agree i. e. they constitute an orthogonal pair. The result of (5.15) remains the same if there are further single states c, $d, .. , f,$ .. on both sides of the matrix element with corresponding occupation numbers as follows

$$\langle N_a, N_b, N_c, .. ,N_f, ..| \boldsymbol{F}^{(1)} \mid N_a +1, N_b -1 , N_c, .. ,N_f, ..\rangle = \sqrt{(N_a +1)}\sqrt{N_b} \cdot f_{ba}^{(1)}$$

or generally (5.16)

$$\langle N_a, .. , N_d, .. ,N_g, ..| \boldsymbol{F}^{(1)} \mid N_a, .. ,N_d +1, .. ,N_g - 1, ..\rangle = \sqrt{(N_d +1)}\sqrt{N_g} \cdot f_{gd}^{(1)}.$$

Thus, in the non-vanishing matrix elements of $\boldsymbol{F}^{(1)}$ at most one occupation number on each side may exceed the corresponding one at most by 1.

5.2 Creation and annihilation operators

We employ a trick which permits to formulate the matrix elements of $\boldsymbol{F}^{(1)}$ in a simple way, and to this end, we introduce the annihilation operator $\boldsymbol{b_f}$ which reduces the boson number. Its definition reads: If the annihilation operator $\boldsymbol{b_f}$ acts on the boson state $\mid N_a, N_b, .., N_f, ..\rangle$ this one is replaced by the normalised and completely symmetric state $\mid N_a, N_b, .., N_f - 1, ..\rangle$ in which the occupation number N_f is reduced to $N_f - 1$. Furthermore the factor $\sqrt{N_f}$ has to be attached i. e.

$$\boldsymbol{b_f} \mid N_a, N_b, .., N_f, ..\rangle = \sqrt{N_f} \mid N_a, N_b, .., N_f - 1, ..\rangle. \tag{5.17}$$

We form the following matrix element

$$\langle N_a, N_b, .., N_f - 1, .. \mid \boldsymbol{b_f} \mid N_a, N_b, .., N_f, ..\rangle =$$

$$\langle N_a, N_b, .., N_f - 1, .. \mid N_a, N_b, .., N_f - 1, ..\rangle \cdot \sqrt{N_f} = \sqrt{N_f}. \tag{5.18}$$

It is known that the left hand side of the first expression in (5.18) represents the conjugate complex form of the given state function. Now we go to the conjugate complex of the matrix element (5.18). We postulate that there exists an operator, say $\boldsymbol{b_f^+}$ which generates just this conjugate matrix element if it is placed between the states $\langle N_a, N_b, .., N_f , .. \mid$ and $\mid N_a, N_b, .., N_f - 1, ..\rangle$ i. e.

$$\sqrt{N_f} = \langle N_a, N_b, .., N_f - 1, .. \mid \boldsymbol{b_f} \mid N_a, N_b, .., N_f, ..\rangle^* =$$

$$\langle N_a, N_b, .., N_f , .. \mid \boldsymbol{b_f^+} \mid N_a, N_b, .., N_f - 1, ..\rangle. \tag{5.19}$$

We learn from (5.19) that the so-called adjoint operator $\boldsymbol{b_f^+}$ has the following effect:

$$\boldsymbol{b_f^+} \mid N_a, N_b, .., N_f - 1, ..\rangle = \sqrt{N_f} \mid N_a, N_b, .., N_f, ..\rangle. \qquad (5.20)$$

The operator $\boldsymbol{b_f^+}$ raises the occupation number by 1 and attaches the root of the new occupation number as a factor. Consequently, $\boldsymbol{b_f^+}$ is named creation operator.

Let's act the operators $\boldsymbol{b_f^+}$ and $\boldsymbol{b_p}$ ($f \neq p$) one after the other:

$$\boldsymbol{b_f^+ b_p} \mid N_{a\,..}, N_f, .., N_p, ..\rangle =$$

$$\boldsymbol{b_f^+} \sqrt{N_p} \mid N_{a\,..}, N_f, .., N_p - 1, ..\rangle = \sqrt{(N_f + 1)}\sqrt{N_p} \mid N_{a\,..}, N_f + 1, .., N_p - 1, ..\rangle, \quad (5.21)$$

or $\boldsymbol{b_p\, b_f^+} \mid N_{a\,..}, N_f, .., N_p, ..\rangle = \sqrt{(N_f + 1)}\sqrt{N_p} \mid N_{a\,..}, N_f + 1, .., N_p - 1, ..\rangle.$ (5.22)

If both operators act on the same single boson state ($f = p$) we have

$$\boldsymbol{b_f^+\, b_f} \mid N_{a\,..}, N_f, ..\rangle = \boldsymbol{b_f^+} \sqrt{N_f} \mid N_{a,\,..}, N_f - 1, ..\rangle = N_f \mid N_{a,\,..}, N_f, ..\rangle. \qquad (5.23)$$

In (5.23) the total state remains unchanged but $\boldsymbol{b_f^+\, b_f}$ attaches the factor N_f i. e. the effect of the operator

$$\boldsymbol{N_f} \equiv \boldsymbol{b_f^+\, b_f} \qquad (5.24)$$

is simply multiplicative.

5.3 Single- and two-boson operators represented by creation and annihilation operators

First, we will show that calculating a matrix element we can replace the single-boson operator $\boldsymbol{F^{(1)}}$ by a combination of creation and annihilation operators in the following way

$$\boldsymbol{F^{(1)}} = \sum_{p,q} f_{pq}^{(1)} \cdot \boldsymbol{b_p^+\, b_q}. \qquad (5.25)$$

The integrals $f_{pq}^{(1)}$ are defined in (5.8) and (5.14). First, we check the relation (5.25) for a diagonal matrix element. For $p = q = d$ we obtain according to (5.24)

$$\langle\, N_a, .. N_d, .. \mid f_{dd}^{(1)} \cdot \boldsymbol{b_d^+\, b_d} \mid N_a, .. N_d, ..\,\rangle = f_{dd}^{(1)} \cdot N_d. \qquad (5.26)$$

Terms with $p \neq q$ vanish because in this case the resulting state function $|N_a,.. ,N_p + 1, .. ,N_q - 1, ..\,\rangle$ is orthogonal to $\langle N_a, .. ,N_p, .. ,N_q, .. |$. Consequently, in order to form the diagonal matrix element of $\boldsymbol{F^{(1)}}$ we have to sum up the terms of the type (5.26). The result is in agreement with the directly calculated expression (5.10).

We now turn to the off diagonal matrix element. It reads

$$\sum_{pq} \langle N_a, .. ,N_d, .. ,N_g, .. \mid f_{pq}^{(1)} \cdot \boldsymbol{b_p^+\, b_q} \mid N_a, .. ,N_d + 1, .. ,N_g - 1, ..\,\rangle \qquad (5.27)$$

Only the term with $p = g$ and $q = d$ differs from zero and yields the expression $f_{dg}^{(1)}\sqrt{(N_d + 1)}\sqrt{N_g}$ in agreement with (5.16). Namely, if $\boldsymbol{b}$-operators with other p

and q values act on the right hand side of (5.27) the resulting state is orthogonal to the one on the left-hand side.

In matrix elements lying farther from the diagonal, the operator $\boldsymbol{b_p^+}\,\boldsymbol{b_q}$ is not able to bring both sides in agreement. Therefore, such elements vanish as we have realised directly in section 5.1. Thus, the assertion (5.25) is proved.

Creation and annihilation operators are also employed in order to represent matrix elements of two-boson operators. The original form of these operators is given by the equation (5.2)

$$\boldsymbol{F}^{(2)} = \sum_{i>j=1}^{N} \boldsymbol{f}^{(2)}(\xi_i,\xi_j).$$

Similar consideration as those for $\boldsymbol{F}^{(1)}$ lead to the following expression for the operator $\boldsymbol{F}^{(2)}$

$$\boldsymbol{F}^{(2)} = ½\cdot\sum_{dgpq} f^{(2)}_{dgpq}\, \boldsymbol{b_d^+}\, \boldsymbol{b_g^+}\, \boldsymbol{b_p}\, \boldsymbol{b_q} \qquad (5.28)$$

with $\quad f^{(2)}_{dgpq} = \int \psi^*_d(\xi)\cdot\psi^*_g(\zeta)\cdot f^{(2)}(\xi,\zeta)\ \psi_p(\xi)\cdot\psi_q(\zeta)\ d\xi\ d\zeta.$

Correspondingly to (5.25), in (5.28) the sum extends over the single states. Diagonal matrix elements read now

$$\langle\, N_a, ..\,, N_p, ..\,, N_q, ..|\, \boldsymbol{F}^{(2)}\, |\, N_a, ..\,, N_p, ..\,, N_q, ..\,\rangle = ½\sum_{pq} f^{(2)}_{pqpq}\cdot N_p\cdot N_q =$$

$$½\sum_r f^{(2)}_{rrrr}\, N_r^2 + \sum_{p<q} f^{(2)}_{pqpq}\, N_p\, N_q. \qquad (5.29)$$

The remaining not vanishing matrix elements of the two-boson operator have the following form

$$\langle\, N_a, .., N_d, .., N_g, .., N_p, .., N_q, ..|\, \boldsymbol{F}^{(2)}\, |\, N_a, .., N_d\text{-}1, .., N_g\text{-}1, .., N_p\text{+}1, .., N_q\text{+}1, ..\rangle =$$

$$½\, f^{(2)}_{dgpq}\, \sqrt{N_d}\, \sqrt{N_g}\, \sqrt{(N_p + 1)}\, \sqrt{(N_q + 1)}. \qquad (5.30)$$

In the broader sense they can be named off diagonal.

We have found matrix elements of $\boldsymbol{F}^{(1)}$ and $\boldsymbol{F}^{(2)}$ on the “primitive“ basis of total states (4.15). **Because every state of the seniority scheme (3.8) can be built with “primitive“ states (section 4.1) the operator representations (5.25) and (5.28) hold also in the seniority scheme**. Although the annihilation and creation operators were introduced as a formal aid, they have a central importance in the interacting boson model.

We put together further properties of the operators $\boldsymbol{b_p^+}$ and $\boldsymbol{b_q}$. According to (5.24) the equation

$$\boldsymbol{b_p^+}\, \boldsymbol{b_p} = \boldsymbol{N_p} \text{ holds.}$$

Analogously one finds $\quad \boldsymbol{b_p}\, \boldsymbol{b_p^+} = \boldsymbol{N_p} + 1. \qquad (5.31)$

Subtracting the equation (5.24) from (5.31) one obtains the commutation rules for the operators $\boldsymbol{b_p}$ and $\boldsymbol{b_p^+}$:

$$\boldsymbol{b_p\, b_p^+ - b_p^+\, b_p} = 1. \tag{5.32}$$

In pairs of operators with different indices p and q the operators can be interchanged according to (5.21) and (5.22) which yields the following relation

$$\boldsymbol{b_p\, b_q^+ - b_q^+\, b_p} = 0. \qquad (\, p \neq q \,) \tag{5.33}$$

We sum up

$$\boldsymbol{b_p\, b_q^+ - b_q^+\, b_p} = \delta_{pq}. \tag{5.34}$$

If there are only annihilation or creation operators the following relations hold

$$\begin{aligned} &\boldsymbol{b_p\, b_q - b_q\, b_p} = 0, \\ &\boldsymbol{b_p^+\, b_q^+ - b_q^+\, b_p^+} = 0. \end{aligned} \tag{5.35}$$

We write the commutation rules for boson operators especially for s- and d-operators :

$$\begin{aligned} &[\boldsymbol{d_\mu\, ,\, d^+_\nu}] \equiv \boldsymbol{d_\mu\, d^+_\nu - d^+_\nu\, d_\mu} = \delta_{\mu\nu} \\ &[\boldsymbol{s\, ,\, s^+}] \equiv \boldsymbol{ss^+ - s^+\, s} = 1 \end{aligned} \tag{5.36}$$

All other commutators of these operators vanish.

Following the corresponding methods of the quantum electrodynamics, the use of the creation and annihilation operators occasionally is named second quantisation. It is also used for the description of the harmonic oscillator or in the shell model of the atomic nucleus.

6 Applications of the creation and annihilation operators

In the first section the formation of “primitive“ many-boson states and states with defined angular momentum will be described in terms of creation operators. In the following section we will show how the number of bosons can be increased with the aid of operators leaving the seniority unchanged. In section 6.3 the annihilation operator is modified in order to fulfil the rules of angular momentum coupling. The inverse of the pair-generating operator is treated in section 6.4. In the end the boson counting operators are put together.

6.1 Many-boson configurations represented by operators

The creation operators can act not only on fully symmetric many-boson states but also on the so-called vacuum state $|\rangle$ which is realised in the case the protons and neutrons fill closed shells and there are no active bosons. Starting from the vacuum state we now construct “primitive“ states i. e. **states without defined angular momentum**.

We employ a creation operator $\boldsymbol{b_a}^+$ whose index a represents the angular momentum of the single state and its projection. It raises the occupation number N_a of this state according to (5.20) as follows

$$\boldsymbol{b_a}^+| N_a - 1 \rangle = \sqrt{N_a} \,| N_a \rangle. \tag{6.1}$$

The effect of $\boldsymbol{b_a}^+$ on the vacuum state $|\rangle$ is defined like this

$$\boldsymbol{b_a}^+| \rangle = | N_a = 1 \rangle. \tag{6.2}$$

If a number N_a of operators $\boldsymbol{b_a}^+$ acts together on $|\rangle$ one obtains the following symmetric many-boson state

$$(\boldsymbol{b_a}^+)^{N_a}| \rangle = \sqrt{(N_a!)}\, | N_a \rangle. \tag{6.3}$$

A normalised, symmetric state with occupation numbers N_a, N_b, .. ,N_f , .. reads therefore

$$(N_a!N_b! \,..\, N_f!)^{-½}\, (\boldsymbol{b_a}^+)^{N_a}(\boldsymbol{b_b}^+)^{N_b} \,..\, (\boldsymbol{b_f}^+)^{N_f}\, | \rangle = | N_a\, N_b \,..\, N_f \rangle. \tag{6.4}$$

The symmetry of the state (6.4) comes from the definition of the creation operator (section 5.2). Any sequence of the operators on the left-hand side of (6.4) can be chosen because they commute according to (5.35).

We now turn to **many-boson states with defined angular momentum**. From states with a few bosons, we learn how the creation operators have to be arranged.

First, we will look at two d-bosons which are coupled to J. According to (4.4) we have

$$| d^2 J M \rangle = (1/\sqrt{2})\cdot\Sigma_{m\neq m'} (2\ m\ 2\ m' | J\ M)\cdot| d\ m\ d\ m' \rangle +$$

$$(2\ m\ 2\ m\ | J\ M)\cdot| d\ m\ d\ m \rangle\cdot \delta_{M\ \text{even}}.$$

With $| d\ m\ d\ m' \rangle = \boldsymbol{d_m^+}\ \boldsymbol{d_{m'}^+} | \rangle$ (for $m \neq m'$) and $| d\ m\ d\ m \rangle = (1/\sqrt{2})\ (\boldsymbol{d_m^+})^2 |\rangle$ we obtain

$$| [d \times d]^{(J)}{}_M \rangle \equiv | d^2 J M \rangle =$$

$$(1/\sqrt{2})\cdot\Sigma_{mm'} (2\ m\ 2\ m' | J\ M)\cdot\ \boldsymbol{d_m^+}\ \boldsymbol{d_{m'}^+} | \rangle \equiv (1/\sqrt{2})\cdot[\boldsymbol{d^+} \times \boldsymbol{d^+}]^{(J)}{}_M |\rangle \qquad (6.5)$$

Thus, in order to construct the symmetric state with defined angular momentum the creation operators have to be coupled in the same way as the single states. A normalisation factor arises (here it is $1/\sqrt{2}$). Analogously to (A2.7) the operator relation

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)}{}_M = 0 \quad \text{for odd } J \qquad (6.6)$$

holds because the operators $\boldsymbol{d}^+{}_{\boldsymbol{m}}$ and $\boldsymbol{d}^+{}_{\boldsymbol{m'}}$ can be interchanged in the same way as the state functions in (A2.2).

We now look into the combination of three d-bosons. According to (4.6) we have

$$| d^3 J_0 J M \rangle = A\cdot \Sigma_{M_0 m''} (J_0\ M_0\ 2\ m'' | J\ M)\cdot \qquad (6.7)$$

$$\Sigma_{mm'} (2m\ 2\ m' | J_0\ M_0)\cdot(2A_p(m,m',m''))^{-1} | d\ m\ d\ m'\ d\ m'' \rangle.$$

The expressions for A_p are given in section 4.1. The “primitive“ states can be written as follows

$$\boldsymbol{d_m^+}\ \boldsymbol{d_{m'}^+}\ \boldsymbol{d_{m''}^+} |\rangle = | d\ m\ d\ m'\ d\ m'' \rangle \qquad \text{with } m \neq m' \neq m'' \neq m,$$

$$(1/\sqrt{2})\ (\boldsymbol{d_m^+})^2\ \boldsymbol{d_{m''}^+} |\rangle = | d\ m\ d\ m\ d\ m'' \rangle \qquad \text{with } m \neq m''$$

and $\qquad (1/\sqrt{(3!)})\ (\boldsymbol{d_m^+})^3 = | d\ m\ d\ m\ d\ m\rangle.$

Using the function $A_p(m,m',m'')$ we summarise

$$\sqrt{6}\cdot A_p(m,m',m'')\ \boldsymbol{d_m^+}\ \boldsymbol{d_{m'}^+}\ \boldsymbol{d_{m''}^+} |\rangle = | d\ m\ d\ m'\ d\ m'' \rangle \qquad (6.8)$$

and insert the result in (6.7).

$$| [[d \times d]^{(J_0)} \times d]^{(J)}{}_M \rangle \equiv | d^3 J_0 J M \rangle =$$

$$A\cdot\sqrt{(3/2)}\ \Sigma_{M_0 m''} (J_0\ M_0\ 2\ m'' | J\ M)\cdot\Sigma_{mm'} (2\ m\ 2\ m' | J_0\ M_0)\cdot\ \boldsymbol{d_m^+}\ \boldsymbol{d_{m'}^+}\ \boldsymbol{d_{m''}^+} |\rangle =$$

$$A\cdot\sqrt{(3/2)}\cdot [[\boldsymbol{d^+} \times \boldsymbol{d^+}]^{(J_0)} \times \boldsymbol{d^+}]^{(J)}{}_M |\rangle. \qquad (6.9)$$

Again, the creation operators have to be coupled in the same way as the single boson states. Here the normalisation factor reads $A\surd(3/2)$. This coupling rule holds generally for symmetric states with defined angular momentum. Operators like $\boldsymbol{d}^{+}$ which obey the angular coupling rules are named tensor operators.

6.2 Generating boson pairs with total angular momentum zero

Here we deal again with many d-boson states in the seniority scheme introduced in (3.8). These states are built with boson pairs having J = 0 which are coupled to a state with τ d-bosons which is characterised by n_Δ, J and M (section 3.3). We investigate how this coupling procedure forming symmetric states can be represented by operators. We symbolise the operator which generates the normalised state of τ boson by $[(\boldsymbol{d}^{+})^{\tau}{}_{n\Delta}]^{(J)}{}_{M}$ i. e.

$$| n_d = \tau, \tau \; n_\Delta \; J \; M \rangle = [(\boldsymbol{d}^{+})^{\tau}{}_{n\Delta}]^{(J)}{}_{M} |\rangle. \qquad (6.10)$$

According to the statements of section 6.1 a J=0-boson pair coupled up to the state (6.10) is described in the following way

$$\begin{aligned} | n_d = \tau + 2, \tau \; n_\Delta \; J \; M \rangle &= A' [[\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(0)} \times [(\boldsymbol{d}^{+})^{\tau}{}_{n\Delta}]^{(J)}]^{(J)}{}_{M} |\rangle = \\ &A' [\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(0)} \cdot [(\boldsymbol{d}^{+})^{\tau}{}_{n\Delta}]^{(J)}{}_{M} |\rangle = \\ &A' (1/\surd 5) \surd 5 [\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(0)} \cdot | n_d = \tau, \tau \; n_\Delta \; J \; M \rangle = \\ &A' (1/\surd 5) \boldsymbol{P}^{+} | n_d = \tau, \tau \; n_\Delta \; J \; M \rangle. \qquad (6.11) \end{aligned}$$

In the last line of (6.11) the so-called pair creation operator $\boldsymbol{P}^{+}$ has been introduced which is defined as follows

$$\boldsymbol{P}^{+} = \Sigma_{\mu} (-1)^{\mu} \boldsymbol{d}^{+}{}_{\mu} \, \boldsymbol{d}^{+}{}_{-\mu} = \surd 5 \; \Sigma_{\mu} (1/\surd 5) \, (-1)^{\mu} \boldsymbol{d}^{+}{}_{\mu} \, \boldsymbol{d}^{+}{}_{-\mu} = \surd 5 [\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(0)}. \quad (6.12)$$

Several authors attach the factor ½ to the right hand side of (6.12). Sometimes the symbol $\boldsymbol{P}^{+} = \boldsymbol{d}^{+} \bullet \boldsymbol{d}^{+}$ is used.

We now generalise the relation (6.11) starting from the state $|n_d - 2, \tau \; n_\Delta \; J \; M \rangle$ in which the seniority τ may fall below its maximal value $n_d - 2$ i. e. $n_d - 2 \geq \tau$, on condition that $n_d - 2 - \tau$ is even. In chapter 10 the following relation will be shown (see 10.15)

$$\begin{aligned} | n_d \; \tau \; n_\Delta \; J \; M \rangle &= (n_d(n_d + 3) - \tau(\tau + 3))^{-½} \cdot \boldsymbol{P}^{+} |n_d - 2, \tau \; n_\Delta \; J \; M \rangle = \\ &((n_d + \tau + 3)(n_d - \tau))^{-½} \cdot \boldsymbol{P}^{+} |n_d - 2, \tau \; n_\Delta \; J \; M \rangle. \qquad (6.13) \end{aligned}$$

Replacing n_d by the expression τ + 2 in (6.13) and comparing with (6.11) we obtain A' = $1/\surd 2$. Analogously to (6.13) the relation

$$\begin{aligned} &|n_d - 2, \tau \; n_\Delta \; J \; M \rangle = \\ &((n_d + \tau + 3 - 2)(n_d - \tau - 2))^{-½} \cdot \boldsymbol{P}^{+} |n_d - 4, \tau \; n_\Delta \; J \; M \rangle \qquad (6.14) \end{aligned}$$

holds which we insert in (6.13). We continue this procedure until the boson number in the state on the right hand side is τ and this state has the form (6.10). Finally, one obtains

$$| n_d\, \tau\;\; n_\Delta\; J\, M \rangle = A_{n_d\tau}\, (\boldsymbol{P}^{+})^{(n_d\, -\, \tau)/2}\; | \,\tau\; \tau\;\; n_\Delta\; J\, M \rangle \tag{6.15}$$

with $\qquad A_{n_d\tau} = ((n_d + \tau + 3)(n_d + \tau + 3 - 2)\; ..\; (2\,\tau + 5)\cdot(\, n_d - \tau)(\, n_d - \tau - 2)\cdot..2)^{-½}$

$$= ((2\,\tau + 3)!!/((n_d + \tau + 3)!!\; (n_d - \tau)!!))^{½}. \tag{6.16}$$

The symbol $n!!$ stands for $n(n - 2)(n - 4)\cdot..\cdot\, 2$ or 1.

To complete the picture we write the states of the seniority scheme including the s-bosons whose number is n_s. According to (6.3) we have

$$(\boldsymbol{s}^{+})^{n_s}\, |\rangle = \sqrt{(n_s!)}\; |\; n_s \rangle. \tag{6.17}$$

We let the operator $(\boldsymbol{s}^{+})^{n_s}\, /\sqrt{(n_s!)}$ act on (6.15) creating a normalised and completely symmetric state as follows

$$| \,n_s\, n_d\, \tau\, n_\Delta\, J\, M \rangle =$$

$$((2\tau + 3)!!/(n_s!(n_d + \tau + 3)!!(n_d - \tau)!!))^{½}\, (\boldsymbol{s}^{+})^{n_s}\, (\boldsymbol{P}^{+})^{(n_d\, -\, \tau)/2}\; | \,\tau\, \tau\, n_\Delta\, J\, M \rangle. \tag{6.18}$$

The structure of the d-boson states $| \,\tau\, \tau\, n_\Delta\, J\, M \rangle$ with maximal seniority may be complex but seldom they have to be formulated explicitly. For the most part it is sufficient to give the so-called “coefficients of fractional parentage“ which permit a representation by states with τ - 1 d-bosons. In this report we don’t deal with this method.

6.3 Tensor operators annihilating bosons

In section 6.1 we have shown that the creation operators $\boldsymbol{d}^{+}_{\mu}$ are components of the tensor operator $\boldsymbol{d}^{+}$ i. e. they obey the rules of angular momentum coupling using Clebsch-Gordan coefficients. This property, which correlates with a special behaviour on the occasion of rotations of the co-ordinate system is very important for treating matrix elements. Owing to the Wigner-Eckart theorem (appendix A5) the matrix elements of tensor operators can be simplified heavily by writing them as a product of a reduced matrix element and a Clebsch-Gordan coefficient.

It can be shown that the annihilation operators $\boldsymbol{d}_{\mu}$ cannot be coupled vectorially but operators of the form

$$\boldsymbol{d}^{\sim}_{\mu} = (-1)^{\mu}\cdot\boldsymbol{d}_{-\mu} \tag{6.19}$$

have this property. They are named modified hermitian adjoint operators. Thus, we maintain that the operator $\boldsymbol{d}^{\sim}$ whose components are defined in (6.19) is a tensor operator. We verify this statement on two simple examples.

If the claim holds, an operator expression such as $[\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{+}]^{(0)}$ can be built which we let act on the vacuum state $|\rangle$. In analogy with the operator $\boldsymbol{P}^{+}$ ((6.12)) a state with angular momentum 0 must result which contains no bosons because the creation and the annihilation operator cancel out in every term. So we expect

$$[\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{+}]^{(0)} |\rangle = B\cdot|\rangle \qquad (6.20)$$

with a constant B. Employing (A1.1) and (A1.16) we remodel the left hand side of (6.20)

$$[\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{+}]^{(0)} |\rangle = \Sigma_{\mu} (2\ \mu\ 2, -\mu \mid 0\ 0)\ \boldsymbol{d}^{\sim}_{\boldsymbol{\mu}}\, \boldsymbol{d}^{+}_{\boldsymbol{-\mu}} |\rangle =$$

$$\Sigma_{\mu} (-1)^{(2-\mu)} (5)^{-½} (-1)^{\mu}\, \boldsymbol{d}_{\boldsymbol{-\mu}}\, \boldsymbol{d}^{+}_{\boldsymbol{-\mu}} |\rangle = \Sigma_{\mu} (5)^{-½}\, \boldsymbol{d}_{\boldsymbol{-\mu}}| d_{-\mu}\rangle = (5)^{-½}\, \Sigma_{\mu} |\rangle = 5(5)^{-½}|\rangle. \qquad (6.21)$$

Therefore the relation (6.20) is satisfied with $B = \sqrt{5}$.

In the second example the operator $\boldsymbol{d}^{\sim}$ is coupled up to the state $[\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(J_0)}$ in such a way that the angular momentum 2 with the component ν results. Because in every term one annihilating and two creating operators are opposite we expect that the resulting state consists in one boson in the state $| d_{\nu} \rangle$ i. e.

$$[\boldsymbol{d}^{\sim} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(J_0)}]^{(J=2)}{}_{\nu} |\rangle = B'\cdot \boldsymbol{d}^{+}_{\nu}|\rangle. \qquad (6.22)$$

J_0 is even because the original two-boson state is symmetric. We write the left-hand side of (6.22) as

$$[\boldsymbol{d}^{\sim} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(J_0)}]^{(J=2)}{}_{\nu} |\rangle =$$

$$\Sigma_{M_0\,\mu} (2\ \mu\ J_0\, M_0 \mid 2\ \nu)\ \Sigma_{\mu'\mu''} (2\ \mu'\ 2\ \mu'' \mid J_0\, M_0)\ \boldsymbol{d}^{\sim}_{\boldsymbol{\mu}} \boldsymbol{d}^{+}_{\boldsymbol{\mu'}}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu''}}|\rangle. \qquad (6.23)$$

With (6.19) and the commutation rules (5.36) we obtain

$$(-1)^{-\mu}\, \boldsymbol{d}^{\sim}_{\boldsymbol{\mu}}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu'}}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu''}}|\rangle = \boldsymbol{d}_{\boldsymbol{-\mu}}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu'}}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu''}}|\rangle = (\delta_{\mu'-\mu}\cdot \boldsymbol{d}^{+}_{\boldsymbol{\mu''}} + \boldsymbol{d}^{+}_{\boldsymbol{\mu'}}\, \boldsymbol{d}_{\boldsymbol{-\mu}}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu''}})|\rangle =$$

$$(\delta_{\mu',-\mu}\cdot \boldsymbol{d}^{+}_{\boldsymbol{\mu''}} + \delta_{\mu'',-\mu}\cdot \boldsymbol{d}^{+}_{\boldsymbol{\mu'}} + \boldsymbol{d}^{+}_{\boldsymbol{\mu'}}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu''}}\, \boldsymbol{d}_{\boldsymbol{-\mu}})|\rangle = (\delta_{\mu',-\mu}\cdot \boldsymbol{d}^{+}_{\boldsymbol{\mu''}} + \delta_{\mu'',-\mu}\cdot \boldsymbol{d}^{+}_{\boldsymbol{\mu'}})|\rangle \qquad (6.24)$$

which we insert in (6.23) as follows

$$[\boldsymbol{d}^{\sim} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(J_0)}]^{(J=2)}{}_{\nu} |\rangle =$$

$$\Sigma_{\mu\, M_0\, \mu''} (2\ \mu\ J_0\, M_0 \mid 2\ \nu)(2, -\mu\ 2\ \mu'' \mid J_0\, M_0)(-1)^{\mu}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu''}}|\rangle +$$

$$\Sigma_{\mu\, M_0\, \mu'} (2\ \mu\ J_0\, M_0 \mid 2\ \nu)(2\ \mu'\ 2, -\mu \mid J_0\, M_0)(-1)^{\mu}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu'}} |\rangle = \qquad (6.25)$$

$$2 \cdot\Sigma_{\mu} (\Sigma_{M_0\, \mu'} (2\ \mu\ J_0\, M_0 \mid 2\ \nu)(2, -\mu\ 2\ \mu' \mid J_0\, M_0))(-1)^{\mu}\, \boldsymbol{d}^{+}_{\boldsymbol{\mu'}} |\rangle.$$

We made use of the symmetry of the last Clebsch-Gordan coefficient. This is rewritten using (A1.6) like this

$$(2,-\mu\ 2\ \mu' \mid J_0\ M_0) = (2, -\mu\ J_0\ M_0 \mid 2, -\mu')((2J_0 + 1)/5)^{½}\cdot(-1)^{\mu} =$$

$$(2\ \mu\ J_0\ M_0 \mid 2\ \mu')((2J_0 + 1)/5)^{½}\cdot(-1)^{\mu}\ . \qquad (6.26)$$

Owing to (A1.10) we obtain

$$[\boldsymbol{d}^{\sim} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(J_0)}]^{(J = 2)}{}_{\nu}\,|\rangle = 2\cdot((2J_0 + 1)/5)^{½}\cdot\sum_{\mu'} \delta_{\mu'\nu}\, \boldsymbol{d}^{+}{}_{\mu'}|\rangle =$$

$$2\cdot((2J_0 + 1)/5)^{½}\cdot\boldsymbol{d}^{+}{}_{\nu}|\rangle \qquad (6.27)$$

in agreement with our forecast (6.22). Both examples illustrate that $\boldsymbol{d}^{\sim}$ is a tensor operator.

6.4 Annihilating boson pairs with total angular momentum zero

Analogously to (6.12) we define the pair annihilation operator $\boldsymbol{P}^{-}$ as follows

$$\boldsymbol{P}^{-} = \sum_{\mu}(-1)^{\mu}\cdot\ \boldsymbol{d}_{\mu}\,\boldsymbol{d}_{-\mu} = \surd 5\cdot\sum_{\mu}(-1)^{\mu}\,(5)^{-½}\ \boldsymbol{d}^{\sim}{}_{\mu}\,\boldsymbol{d}^{\sim}{}_{-\mu} = \surd 5\cdot\sum_{\mu}(2\ \mu\ 2, -\mu \mid 0\ 0)\boldsymbol{d}^{\sim}{}_{\mu}\,\boldsymbol{d}^{\sim}{}_{-\mu} =$$

$$\surd 5\cdot[\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(0)}. \qquad (6.28)$$

Several authors attach the factor ½ to the right hand side of (6.28). Sometimes the symbol $\boldsymbol{P}^{-} = \boldsymbol{d}^{\sim}\bullet\boldsymbol{d}^{\sim}$ is used. From (10.16) we have

$$\boldsymbol{P}^{-}\,|n_d\ \tau\ n_{\Delta}\ J\ M\,\rangle = (n_d(n_d + 3) - \tau(\tau + 3))^{½}\ |n_d\ \text{-}2,\ \tau\ n_{\Delta}\ J\ M\,\rangle. \qquad (6.29)$$

The operator

$$(n_d(n_d + 3) - \tau(\tau + 3))^{-½}\cdot\ \boldsymbol{P}^{-} \qquad (6.30)$$

is normalised. If $\boldsymbol{P}^{-}$ acts on the state $|\ \tau\ \tau\ n_{\Delta}\ J\ M\,\rangle$ of the seniority scheme (with $n_d = \tau$), from (6.29) follows

$$\boldsymbol{P}^{-}\,|n_d = \tau,\ \tau\ n_{\Delta}\ J\ M\,\rangle = 0. \qquad (6.31)$$

6.5 Number operators for bosons

We let the operator $\boldsymbol{d}^{+}{}_{\mu}\,\boldsymbol{d}_{\mu}$ (with $2 \geq \mu \geq -2$) act on the “primitive“ state $|\ (d_2)^{n_2}\ (d_1)^{n_1}\ (d_0)^{n_0}\ (d_{-1})^{n_{-1}}\ (d_{-2})^{n_{-2}}\,\rangle \equiv |\ n_2\ n_1\ n_0\ n_{-1}\ n_{-2}\,\rangle$. Owing to (5.23) we obtain

$$\boldsymbol{d}^{+}{}_{\mu}\,\boldsymbol{d}_{\mu}|\ n_2\ n_1\ n_0\ n_{-1}\ n_{-2}\,\rangle = n_{\mu}|\ n_2\ n_1\ n_0\ n_{-1}\ n_{-2}\,\rangle. \qquad (6.32)$$

Therefore the eigenvalue of $\boldsymbol{d}^{+}{}_{\mu}\,\boldsymbol{d}_{\mu}$ is the number n_{μ} of d-bosons in the state μ. Consequently the operator $\sum_{\mu}\boldsymbol{d}^{+}{}_{\mu}\,\boldsymbol{d}_{\mu}$ yields

$$\sum_{\mu}\boldsymbol{d}^{+}{}_{\mu}\,\boldsymbol{d}_{\mu}\,|\ n_2\ n_1\ n_0\ n_{-1}\ n_{-2}\,\rangle = \sum_{\mu}n_{\mu}|\ n_2\ n_1\ n_0\ n_{-1}\ n_{-2}\,\rangle = n_d\,|\ n_2\ n_1\ n_0\ n_{-1}\ n_{-2}\,\rangle. \ (6.33)$$

Thus its eigenvalue is the total number n_d of d-bosons. The operator $\sum_{\mu}\boldsymbol{d}^{+}{}_{\mu}\,\boldsymbol{d}_{\mu}$ can be applied also to states with defined total angular momentum because they can be written as a linear combinations of “primitive“ states with the same number of *d*-bosons (section 4.1). This number operator is characterized by $\boldsymbol{n_d}$,

$$\boldsymbol{n_d} = \sum_\mu \boldsymbol{d}^+_\mu \boldsymbol{d}_\mu \ . \tag{6.34}$$

With the help of $\boldsymbol{d}^\sim_{-\mu}(-1)^\mu = \boldsymbol{d}_\mu$ (6.19) and of (A1.16) we write also

$$\boldsymbol{n_d} = \sqrt{5}\,\sum_\mu(-1)^\mu(5)^{-1/2}\boldsymbol{d}^+_\mu \boldsymbol{d}^\sim_{-\mu} = \sqrt{5}\,\sum_\mu(2\ \mu\ 2, -\mu \mid 0\ 0)\boldsymbol{d}^+_\mu \boldsymbol{d}^\sim_{-\mu} =$$

$$\sqrt{5}\cdot[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}. \tag{6.35}$$

The number operator for *s*-bosons is built correspondingly of creation and annihilation operators for *s*-bosons, $\boldsymbol{s}^+$ and $\boldsymbol{s}$, as follows

$$\boldsymbol{n_s} = \boldsymbol{s}^+ \boldsymbol{s}. \tag{6.36}$$

Its eigenvalue is the number of *s*-bosons, n_s. The operator

$$\boldsymbol{N} = \boldsymbol{n_d} + \boldsymbol{n_s} \tag{6.37}$$

has the eigenvalue $n_s + n_d$ .i. e. the total number of bosons of the collective state.

As mentioned above, the use of creation and annihilation operators for bosons frequently is named second quantisation. An advantage of this method consists in the fact that the matrix elements of the Hamilton operator and of the operators describing the electromagnetic transition can be written completely with the boson operators $\boldsymbol{d}^+_\mu$, $\boldsymbol{d}^\sim_\nu$, $\boldsymbol{s}^+$ and $\boldsymbol{s}$. In a word, not only the operators of the interactions (5.25) and (5.28) but also the boson states are represented by the single boson operators. For the algebraic treatment of these elements the commutation rules (5.36) are applied.

7 The Hamilton operator of the IBM1

In the first section of this chapter, the components of the Hamilton operator will be sketched in. The part with boson-boson interactions is transformed in the second section by coupling the tensor operators. In section 7.3 the Hamiltonian is written explicitly with *d*- and *s*-operators. It must conserve the number of bosons i. e. it must commute with the number operator, which is verified in the fourth section. In the last one the Hamiltonian will be brought in a more compact form.

7.1 The components of the Hamiltonian

In chapter 2 the number N of active bosons in the interacting boson model is determined. Formulating the energy of the system, we don't go into the structure and eigenenergy of the inner part of the nucleus i. e. of the closed shell (core). The kinetic energy $T(i)$ and the potential energy $U(i)$ of an active boson have the character of operators and constitute together the Hamilton operator $\boldsymbol{H}^{(1)}$ of a single state $| b_{lm} \rangle$. It generates the eigenenergy ε_{lm} owing to the equation

$$(\boldsymbol{T}^{(1)} + \boldsymbol{U}^{(1)})| b_{lm} \rangle = \boldsymbol{H}^{(1)} | b_{lm} \rangle = \varepsilon_{lm} \; | b_{lm} \rangle \tag{7.1}$$

The singe-boson states $| b_{lm} \rangle$ have a defined angular momentum l with the projection m i. e. we suppose that the angular momentum operator commutes with the Hamiltonian $\boldsymbol{H}^{(1)}$. We mentioned that there are only six lm-states in the IBM1.

Because no spatial axis prevails, the single-boson energy ε_{lm} does not depend on *m.* Therefore there are only two energies of this kind, namely ε_s and ε_d . If the bosons were independent of one another, a system of n_s s-bosons and n_d *d*-boson would have the energy $n_s\varepsilon_s + n_d\varepsilon_d$. Consequently the Hamilton operator of the whole system would contain counting operators (section 6.5) in the following way

$$\varepsilon_s \boldsymbol{n_s} + \varepsilon_d \boldsymbol{n_d} \, . \tag{7.2}$$

From (6.34) and (6.36) we see that this operator has the structure of a single-boson operator (5.25) with $f^{(1)}{}_{p\neq q} = 0$ (see section 5.3). Owing to (6.37) the form

$$\varepsilon_s \boldsymbol{N} + (\varepsilon_d - \varepsilon_s)\boldsymbol{n_d} \, . \tag{7.3}$$

can be used also. We now introduce the interaction between the active bosons from which we assume that it takes place in twos. According to (5.2) we have to add a two-boson operator such as

$$\tfrac{1}{2}\sum_{1=i<j}^{N} \boldsymbol{W}(ij) \equiv \boldsymbol{W} \tag{7.4}$$

to the Hamiltonian. From (5.28), we take the corresponding form

$$\tfrac{1}{2}\sum_{i,j=1}^{N} \boldsymbol{W}(ij) = \tfrac{1}{2}\sum_{f\,g\,p,q=1}^{6} \langle\, f\, g \mid \boldsymbol{w} \mid p\; q \,\rangle\; \boldsymbol{b}^{+}_{f}\,\boldsymbol{b}^{+}_{g}\,\boldsymbol{b}_{p}\,\boldsymbol{b}_{q}\;. \tag{7.5}$$

The indices on the right hand side of (7.5) denote the *s*- and five *d*-states.

7.2 The operator of the boson-boson interaction formulated with defined angular momentum

Before we write the expression (7.5) down for the IBM, we convert it into a form containing operator configurations and matrix elements with defined angular momenta. In (7.5) we replace the indices *f*, *g*, .. by pairs of quantum numbers $l_f\, m_f$, $l_g\, m_g$, .. and make the following claim

$$\boldsymbol{W} \equiv \tfrac{1}{2}\sum_{l_f m_f l_g m_g l_p m_p l_q m_q} \langle\, l_f\, m_f\, l_g\, m_g \mid \boldsymbol{w} \mid l_p\, m_p\, l_q\, m_q \,\rangle\; \boldsymbol{b}^{+}_{l_f m_f}\boldsymbol{b}^{+}_{l_g m_g}\,\boldsymbol{b}_{l_p m_p}\,\boldsymbol{b}_{l_q m_q} = \tag{7.6}$$

$$\tfrac{1}{2}\sum_{l_f\, l_g\, l_p\, l_q, J=\text{even}} \langle\, l_f\, l_g\, J \mid \boldsymbol{w} \mid l_p\, l_q\, J \,\rangle\; \sqrt{(2J+1)}\cdot[[\boldsymbol{b}^{+}_{f} \times \boldsymbol{b}^{+}_{g}]^{(J)} \times [\boldsymbol{b}^{\sim}_{p} \times \boldsymbol{b}^{\sim}_{q}]^{(J)}]^{(0)}.$$

We start the proof by adding the factor $\delta_{m_f m_f'}\,\delta_{m_g m_g'}\,\delta_{m_p m_p'}\,\delta_{m_q m_q'}$ on the left hand side of (7.6) and by summing over m_f', m_g', m_p' and m_q' which yields the original expression. It reads

$$\boldsymbol{W} = \tfrac{1}{2}\sum_{l_f\, l_g\, l_p\, l_q} \sum_{m_f\, m_g\, m_p\, m_q\, m_f'\, m_g'\, m_p'\, m_q'} \langle\, l_f\, m_f\, l_g\, m_g \mid \boldsymbol{w} \mid l_p\, m_p\, l_q\, m_q \,\rangle \cdot$$

$$\delta_{m_f m_f'}\,\delta_{m_g m_g'}\,\delta_{m_p m_p'}\,\delta_{m_q m_q'}\; \boldsymbol{b}^{+}_{l_f m_f'}\; \boldsymbol{b}^{+}_{l_g m_g'}\,\boldsymbol{b}_{l_p m_p'}\,\boldsymbol{b}_{l_q m_q'}\;, \tag{7.7}$$

in which the quantum numbers *m* of the ***b***-operators have been supplied with primes. With the help of (A1.15), i. e. of

$$\sum_{JM} (l_f\, m_f\, l_g\, m_g \mid J\, M)(l_f\, m_f'\, l_g\, m_g' \mid J\, M) = \delta_{m_f m_f'}\,\delta_{m_g m_g'} \tag{7.8}$$

and of an analogous expression for *p* and *q* we obtain

$$\boldsymbol{W} = \tfrac{1}{2}\sum_{l_f\, l_g\, l_p\, l_q} \sum_{J\, M\, J''\, M''} \sum_{m_f\, m_g\, m_p\, m_q\, m_f'\, m_g'\, m_p'\, m_q'} \langle\, l_f\, m_f\, l_g\, m_g \mid \boldsymbol{w} \mid l_p\, m_p\, l_q\, m_q \,\rangle \cdot$$

$$(l_f\, m_f\, l_g\, m_g \mid J\, M)\; (l_p\, m_p\, l_q\, m_q \mid J''\, M'')\; (l_f\, m_f'\, l_g\, m_g' \mid J\, M)\; (l_p\, m_p'\, l_q\, m_q' \mid J''\, M'')$$

$$\boldsymbol{b}^{+}_{l_f m_f'}\; \boldsymbol{b}^{+}_{l_g m_g'}\,\boldsymbol{b}_{l_p m_p'}\,\boldsymbol{b}_{l_q m_q'}\;. \tag{7.9}$$

We make use of (A1.1) for the following equations

$$\sum_{m_f'\, m_g'} (l_f\, m_f'\, l_g\, m_g' \mid J\, M)\; \boldsymbol{b}^{+}_{l_f m_f'}\boldsymbol{b}^{+}_{l_g m_g'} = [\boldsymbol{b}^{+}_{f} \times \boldsymbol{b}^{+}_{g}]^{(J)}_{\mathrm{M}} \tag{7.10}$$

and $\quad \sum_{m_p'\, m_q'} (l_p\, m_p'\, l_q\, m_q' \mid J''\, M'')\; \boldsymbol{b}_{l_p m_p'}\,\boldsymbol{b}_{l_q m_q'} =$

$$\sum_{m_p'\, m_q'} (l_p\, m_p'\, l_q\, m_q' \mid J''\, M'')\; (-1)^{-l_p-l_q+m_p'+m_q'}\boldsymbol{b}^{\sim}_{l_p\, -m_p'}\; \boldsymbol{b}^{\sim}_{l_q\, -m_q'} =$$

$$\sum_{m_p'\, m_q'} (l_p, -m_p',\, l_q, -m_q' \mid J'', -M'')\; (-1)^{M''-J''}\; \boldsymbol{b}^{\sim}_{l_p\, -m_p'}\; \boldsymbol{b}^{\sim}_{l_q\, -m_q'} =$$

$$(-1)^{M''-J''}\; [\boldsymbol{b}^{\sim}_{p} \times \boldsymbol{b}^{\sim}_{q}]^{(J'')}_{-M''}\;, \tag{7.11}$$

in which (A1.5) is used and (6.19) is put in the original form $\boldsymbol{b}_{lm} = (-1)^{-l+m}\boldsymbol{b}^{\sim}_{l,\,-m}$. We obtain

$$\boldsymbol{W} = \tfrac{1}{2}\sum_{l_f\, l_g\, l_p\, l_q} \sum_{J\, M\, J''\, M''} \sum_{m_f\, m_g\, m_p\, m_q} \langle\, l_f\, m_f\, l_g\, m_g \,|\, \boldsymbol{w} \,|\, l_p\, m_p\, l_q\, m_q \,\rangle \cdot \tag{7.12}$$

$$(l_f\, m_f\, l_g\, m_g \mid J\, M)\,(l_p\, m_p\, l_q\, m_q \mid J''\, M'')\,(-1)^{M''-J''}\,[\boldsymbol{b}^+{}_f \times \boldsymbol{b}^+{}_g]^{(J)}{}_M [\boldsymbol{b}^\sim{}_p \times \boldsymbol{b}^\sim{}_q]^{(J'')}{}_{-M''}\,.$$

The sum over m_f, m_g, m_p and m_q yields

$$\boldsymbol{W} = \tfrac{1}{2}\sum_{l_f\, l_g\, l_p\, l_q} \sum_{J\, M\, J''\, M''} \langle\, l_f\;\; l_g\, J\, M \,|\, \boldsymbol{w} \,|\, l_p\, l_q\, J''\, M'' \,\rangle \cdot$$

$$(-1)^{M''-J''}[\boldsymbol{b}^+{}_f \times \boldsymbol{b}^+{}_g]^{(J)}{}_M\,[\boldsymbol{b}^\sim{}_p \times \boldsymbol{b}^\sim{}_q]^{(J'')}{}_{-M''}\,. \tag{7.13}$$

The matrix element in (7.13) is constructed in terms of two-boson states with defined total angular momenta J and J'' which, however, must agree. Namely, if we assume that $\boldsymbol{w}$ is a tensor operator with rank 0 (see appendix A5) the Clebsch-Gordan coefficient in (A5.8) vanishes for $(J,M) \neq (J'',M'')$. Moreover, owing to (A5.8), (A1.6) and (A1.16) the matrix element in (7.13) is independent of M. Using (A1.16) we obtain

$$\boldsymbol{W} = \tfrac{1}{2}\sum_{l_f\, l_g\, l_p\, l_q\,,J=\text{even}} \langle\, l_f\;\; l_g\, J \,|\, \boldsymbol{w} \,|\, l_p\, l_q\, J \,\rangle \cdot\sqrt{(2J + 1)}$$

$$[[\boldsymbol{b}^+{}_f \times \boldsymbol{b}^+{}_g]^{(J)} \times [\boldsymbol{b}^\sim{}_p \times \boldsymbol{b}^\sim{}_q]^{(J)}]^{(0)}\,. \tag{7.14}$$

No odd values occur for J if the l's are 0 or 2, under which condition the equation (7.6) is proved.

7.3 The basic form of the Hamilton operator

Now for the operators of the f-, g-, p- and q-states in (7.6) specially we put in the s- and d-operators respectively and make use of the symmetry properties of the matrix elements. The boson-boson interaction operator obtains the following form

$$\boldsymbol{W} = \tfrac{1}{2}\sum_{J=0,2,4} \langle\, d\, d\, J \,|\, \boldsymbol{w} \,|\, d\, d\, J \,\rangle\, \sqrt{(2J + 1)}\; [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} +$$

$$\tfrac{1}{2}\,\langle\, d\, d,J = 2 \,|\, \boldsymbol{w} \,|\, d\, s,J = 2 \,\rangle\sqrt{5}\cdot 2([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times \boldsymbol{d}^\sim \boldsymbol{s}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{d}^+ \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(2)}]^{(0)})+$$

$$\tfrac{1}{2}\,\langle\, d\, d,J = 0 \,|\, \boldsymbol{w} \,|\, s\, s,J = 0 \,\rangle([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} \times \boldsymbol{ss}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{s}^+ \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)}]^{(0)}) +$$

$$\tfrac{1}{2}\,\langle\, d\, s,J = 2 \,|\, \boldsymbol{w} \,|\, d\, s,J = 2 \,\rangle\, \sqrt{5}\; 4[\boldsymbol{d}^+\boldsymbol{s}^+ \times \boldsymbol{d}^\sim\boldsymbol{s}]^{(0)} + \tag{7.15}$$

$$\tfrac{1}{2}\,\langle\, s\, s,J = 0 \,|\, \boldsymbol{w} \,|\, s\, s,J = 0 \,\rangle\; [\boldsymbol{s}^+\boldsymbol{s}^+ \times \boldsymbol{ss}]^{(0)}\,.$$

We introduce usual symbols for the matrix elements as follows

$$\langle\, d\, d\, J \,|\, \boldsymbol{w} \,|\, d\, d\, J \,\rangle = c_J$$

$$\langle\, d\, d,J = 2 \,|\, \boldsymbol{w} \,|\, d\, s,J = 2 \,\rangle\sqrt{(2 \cdot 5)} = v_2$$

$$\langle\, d\, d,J = 0 \,|\, \boldsymbol{w} \,|\, s\, s,J = 0 \,\rangle = v_0 \tag{7.16}$$

$$\langle\, d\, s,J = 2 \,|\, \boldsymbol{w} \,|\, d\, s,J = 2 \,\rangle 2\sqrt{5} = u_2$$

$$\langle\, s\, s,J = 0 \,|\, \boldsymbol{w} \,|\, s\, s,J = 0 \,\rangle = u_0\,.$$

In the following the quantities c_J , v_i and u_i will play the role of parameters. Finally, we put the Hamilton operator together from (7.3), (7.15) and (7.16)

$$\boldsymbol{H} = \varepsilon_s \boldsymbol{N} + (\varepsilon_d - \varepsilon_s) \boldsymbol{n_d} + ½ \textstyle\sum_{J=0,2,4} c_J \sqrt{(2J + 1)}\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} +$$
$$\sqrt{(½)}\, v_2 ([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times \boldsymbol{d}^\sim \boldsymbol{s}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{d}^+ \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(2)}]^{(0)}) +$$
$$½\, v_0 ([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} \times \boldsymbol{ss}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{s}^+ \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)}]^{(0)}) +$$
$$u_2 [\boldsymbol{d}^+\boldsymbol{s}^+ \times \boldsymbol{d}^\sim \boldsymbol{s}]^{(0)} + \tag{7.17}$$
$$½\, u_0 [\boldsymbol{s}^+\boldsymbol{s}^+ \times \boldsymbol{ss}]^{(0)} .$$

We name the representation (7.17) the basic form of the Hamilton operator in the IBM1. In the next section but one we will see that the 9 parameters can be brought to 6 by reducing the field of applicability.

7.4 The conservation of boson number

We expect that collective states with definite energy have also a definite number N of bosons. In quantum mechanical terms this means that the Hamilton operator $\boldsymbol{H}$ and the boson number operator, $\boldsymbol{N} = \boldsymbol{s}^+\boldsymbol{s} + \sum_\mu \boldsymbol{d}^+_\mu \boldsymbol{d}_\mu$ (6.37), commute

$$[\, \boldsymbol{H}, \boldsymbol{N}\,] = 0. \tag{7.18}$$

In order to prove (7.18) we fall back upon the representation (7.5) and write the Hamilton operator this way

$$\boldsymbol{H} = \varepsilon_s \boldsymbol{N} + (\varepsilon_d - \varepsilon_s)\, \boldsymbol{n_d} + ½\textstyle\sum_{f\, g\, p\, q=1}^{6} \langle\, f\, g \mid \boldsymbol{w} \mid p\, q\, \rangle\, \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q}. \tag{7.19}$$

From (6.34) up to (6.37) we take that the number operator reads

$$\boldsymbol{N} = \textstyle\sum_{n=1}^{6} \boldsymbol{b}^+_n \boldsymbol{b_n} . \tag{7.20}$$

In order to see if $\boldsymbol{N}$ commutes with the group of the four b-operators in (7.19) we push $\boldsymbol{N}$ step by step from the left to the right hand side of this group as follows

$$\boldsymbol{N}\, \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} = \textstyle\sum_n \boldsymbol{b}^+_n \boldsymbol{b_n}\, \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} = \sum_n \boldsymbol{b}^+_n (\delta_{n,f} + \boldsymbol{b}^+_f \boldsymbol{b_n})\boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} =$$
$$\boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} + \textstyle\sum_n \boldsymbol{b}^+_f \boldsymbol{b}^+_n \boldsymbol{b_n} \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} = 2\, \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} + \sum_n \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b}^+_n \boldsymbol{b_n} \boldsymbol{b_p} \boldsymbol{b_q} =$$
$$2\, \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} + \textstyle\sum_n \boldsymbol{b}^+_f \boldsymbol{b}^+_g (-\delta_{n,p} + \boldsymbol{b_p} \boldsymbol{b}^+_n)\, \boldsymbol{b_n} \boldsymbol{b_q} = \tag{7.21}$$
$$\boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} + \textstyle\sum_n \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b}^+_n \boldsymbol{b_q} \boldsymbol{b_n} = \sum_n \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} \boldsymbol{b}^+_n \boldsymbol{b_n} = \boldsymbol{b}^+_f \boldsymbol{b}^+_g \boldsymbol{b_p} \boldsymbol{b_q} \boldsymbol{N}.$$

It is not difficult to extend the proof of (7.18) on the other terms in (7.19). Thus, $\boldsymbol{N}$ commutes with $\boldsymbol{H}$, i. e. $\boldsymbol{N}$ is a so-called constant of motion. Provided that the Hamiltonian does not contain s-operators (see chapter 10), implying the condition $v_2 = v_0 = u_2 = u_0 = 0$, the d-number operator $\boldsymbol{n_d} = \sum_\mu \boldsymbol{d}^+_\mu \boldsymbol{d}_\mu$ commutes with $\boldsymbol{H}$. Then the number of d-bosons is definite.

7.5 A simplified form of the Hamilton operator

Since the boson number N is a good quantum number, it's obvious to replace the operators $\boldsymbol{s}$ and $\boldsymbol{s}^+$ as far as possible by $\boldsymbol{N}$. Making use of (5.34) we write

$$[\boldsymbol{s}^+\boldsymbol{s}^+ \times \boldsymbol{ss}]^{(0)} = \boldsymbol{s}^+\boldsymbol{s}^+\boldsymbol{ss} = \boldsymbol{s}^+\boldsymbol{s}\,\boldsymbol{s}^+\boldsymbol{s} - \boldsymbol{s}^+\boldsymbol{s} = \boldsymbol{n_s}(\boldsymbol{n_s} - 1) = (\boldsymbol{N} - \boldsymbol{n_d})(\boldsymbol{N} - \boldsymbol{n_d} - 1). \quad (7.22)$$

In addition, owing to (6.35) we obtain

$$[\boldsymbol{d}^+\boldsymbol{s}^+ \times \boldsymbol{d}^{\sim}\boldsymbol{s}]^{(0)} = [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(0)}\, \boldsymbol{s}^+\boldsymbol{s} = \sqrt{(1/5)}\, \boldsymbol{n_d}\, \boldsymbol{n_s} = \sqrt{(1/5)}\, \boldsymbol{n_d}\, (\boldsymbol{N} - \boldsymbol{n_d}). \quad (7.23)$$

In (7.22) and (7.23) the operator $\boldsymbol{n_d}^2$ appears, for which we give the expression

$$\boldsymbol{n_d}^2 = \boldsymbol{n_d} + \sum_{J=0,2,4} \sqrt{(2J + 1)}[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(J)}]^{(0)}. \quad (7.24)$$

Namely, according to (6.35) we write

$$\boldsymbol{n_d}^2 = 5[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(0)} \cdot [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(0)} = 5[[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(0)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(0)}]^{(0)}. \quad (7.25)$$

From (A4.3) we learn that (7.25) can be rebuilt as follows

$$\boldsymbol{n_d}^2 = 5\sum_J (2J + 1) \{{}^2{}_{2J}\, {}^2{}_{2J}\, {}^0{}_{00}\} \cdot \sum_M (J\, M\, J, -M \mid 0\ 0) \cdot \quad (7.26)$$

$$\sum_\mu (2\ \mu\ 2, M - \mu \mid J\, M) \cdot \sum_{\mu'} (2\ \mu'\ 2, -M - \mu' \mid J, -M) \cdot \boldsymbol{d}^+_\mu\, \boldsymbol{d}^{\sim}_{\mu'}\, \boldsymbol{d}^+_{M-\mu}\, \boldsymbol{d}^{\sim}_{-M-\mu'}.$$

We employ (6.19) and the commutation rules (5.36) and obtain

$$\boldsymbol{d}^+_\mu\, \boldsymbol{d}^{\sim}_{\mu'}\, \boldsymbol{d}^+_{M-\mu}\, \boldsymbol{d}^{\sim}_{-M-\mu'} = (-1)^{\mu'} \delta_{-\mu', M-\mu}\, \boldsymbol{d}^+_\mu\, \boldsymbol{d}^{\sim}_{-M-\mu'} + \boldsymbol{d}^+_\mu\, \boldsymbol{d}^+_{M-\mu}\, \boldsymbol{d}^{\sim}_{\mu'}\, \boldsymbol{d}^{\sim}_{-M-\mu'}. \quad (7.27)$$

This leads the expression (7.26) to fall into two parts. Because J is an integer, from (A4.2), (A3.12) and (A1.16) we obtain

$$\{{}^2{}_{2J}\, {}^2{}_{2J}\, {}^0{}_{00}\} = (1/5)/\sqrt{(2J + 1)} \text{ and } (J\, M\, J, -M \mid 0\ 0) = (-1)^{J-M}/\sqrt{(2J + 1)}\,. \quad (7.28)$$

The first expression of (7.26) reads now

$$\sum_{J=0}^4 (-1)^J \sum_\mu \boldsymbol{d}^+_{\boldsymbol{\mu}}\, \boldsymbol{d_\mu} \sum_M (2\ \mu\ 2, M - \mu \mid J\, M)^2 = \quad (7.29)$$

$$\sum_{J=0}^4 (-1)^J \sum_\mu \boldsymbol{d}^+_{\boldsymbol{\mu}}\, \boldsymbol{d_\mu}\, (2J + 1)/5 = \sum_\mu \boldsymbol{d}^+_{\boldsymbol{\mu}}\, \boldsymbol{d_\mu}\, (1 - 3 + 5 - 7 + 9)/5 = \boldsymbol{n_d}$$

where (A1.11) has been applied. With the aid of (A4.1) and (A1.3), we bring the second expression of (7.26) in the following form

$$\sum_{J=0,2,4} \sqrt{(2J + 1)}[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(J)}]^{(0)}. \quad (7.30)$$

Owing to (6.6), the quantity J is even. With that, the equation (7.24) is proven. We now insert (7.22) up to (7.24) in (7.17) and obtain

$$\boldsymbol{H} = \varepsilon_s N + \tfrac{1}{2}u_0 N(N-1) + \varepsilon\, \boldsymbol{n_d} +$$
$$\tfrac{1}{2}\textstyle\sum_{J=0,2,4} c_J'\sqrt{(2J+1)}\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} +$$
$$\sqrt{(\tfrac{1}{2})}\, v_2\, ([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times \boldsymbol{d}^\sim \boldsymbol{s}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{d}^+ \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(2)}]^{(0)}) + \qquad (7.31)$$
$$\tfrac{1}{2}\, v_0\, ([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} \times \boldsymbol{ss}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{s}^+ \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)}]^{(0)})$$

with $\varepsilon = \varepsilon_d - \varepsilon_s + (u_2/\sqrt{5} - u_0)(N-1)$,

$$c_J' = c_J + u_0 - 2u_2/\sqrt{5}. \qquad (7.32)$$

Since the operators $\boldsymbol{H}$ and $\boldsymbol{N}$ commute (7.21), the operator $\boldsymbol{N}$ has been replaced by the number N. The simplified form (7.31) of the Hamilton operator contains one free parameter less than the basic form (7.17). However, if only the spectrum of a single nucleus has to be investigated neglecting the binding energy relative to other nuclei one can disregard the first two terms in $\boldsymbol{H}$ (7.31). They contribute the same amount to every energy level. The following 6 parameters are left over

$$\varepsilon,\ c_0',\ c_2',\ c_4',\ v_0 \text{ and } v_2. \qquad (7.33)$$

From (7.32) we see that these parameters don't depend directly from one another because there are enough parameters in (7.17).

All d- and s-operators in (7.17) and (7.31) couple to the angular momentum 0. Therefore they don't prefer a spatial axis and are insensitive to rotations of the co-ordinate system. Consequently, we expect that $\boldsymbol{H}$ commutes with the components of the angular momentum operator $\underline{\boldsymbol{J}}$ which will be treated in the next chapter.

8 The angular momentum operator of the IBM1

In the first section the commutation properties of the quantum mechanical angular momentum operators and their eigenvalues will be put together. In the second section the angular momentum operator of the IBM1 is introduced and its properties are checked. Finally, the commutator of the Hamiltonian and the angular momentum operator is investigated.

8.1 The angular momentum operator in quantum mechanics

Introducing the spin of particles quantum mechanics postulates spin operators $\boldsymbol{J_x}$, $\boldsymbol{J_y}$ and $\boldsymbol{J_z}$ which satisfy the same commutation relations as the angular momentum operators $\boldsymbol{L_x} = \boldsymbol{y}\, \boldsymbol{p_z} - \boldsymbol{z}\, \boldsymbol{p_y}$, $\boldsymbol{L_y}$ and $\boldsymbol{L_z}$. Their commutation rules read

$$[\, \boldsymbol{J_x} , \boldsymbol{J_y} \,] = i\hbar \boldsymbol{J_z} \quad \text{with cyclic permutations.} \tag{8.1}$$

From these one derives the existence of quantum states $|\, j\ m \,\rangle$ where j is an integer or half integer and m is its z-component. The following eigenvalue equations hold

$$\boldsymbol{J_z} \,|\, j\ m \,\rangle = \hbar m \,|\, j\ m \,\rangle,$$

$$\boldsymbol{J}^{\boldsymbol{2}} \,|\, j\ m \,\rangle = (\boldsymbol{J_x}^2 + \boldsymbol{J_y}^2 + \boldsymbol{J_z}^2) \,|\, j\ m \,\rangle = \hbar^2\, j(j + 1) \,|\, j\ m \,\rangle\ . \tag{8.2}$$

We define new spin- or angular momentum operators which differ in the factor $1/\hbar$ as follows

$$\boldsymbol{J_x}(\text{new}) = \boldsymbol{J_x}(\text{old})/\hbar \quad \text{etc.} \tag{8.3}$$

We rewrite equations (8.1) and (8.2) in terms of the new operators like this

$$[\, \boldsymbol{J_x} , \boldsymbol{J_y} \,] = i\boldsymbol{J_z} \quad \text{with cyclic permutations,} \tag{8.4}$$

$$\boldsymbol{J_z} \,|\, j\ m \,\rangle = m \,|\, j\ m \,\rangle, \tag{8.5}$$

$$\boldsymbol{J}^{\boldsymbol{2}} \,|\, j\ m \,\rangle = (\boldsymbol{J_x}^2 + \boldsymbol{J_y}^2 + \boldsymbol{J_z}^2) \,|\, j\ m \,\rangle = j\,(j + 1) \,|\, j\ m \,\rangle.$$

From now on, we keep the new J-operators. The frequently used “ladder“ operators read

$$\boldsymbol{J_1} = -(\boldsymbol{J_x} + i\boldsymbol{J_y})/\sqrt{2},$$

$$\boldsymbol{J_{-1}} = (\boldsymbol{J_x} - i\boldsymbol{J_y})/\sqrt{2}, \tag{8.6}$$

$$\boldsymbol{J_0} = \boldsymbol{J_z}\ .$$

They satisfy the following commutation relations

$$[\, \boldsymbol{J_{-1}}, \boldsymbol{J_0}\,] = \boldsymbol{J_{-1}},$$

$$[\, \boldsymbol{J_{-1}}, \boldsymbol{J_1}\,] = \boldsymbol{J_0}, \qquad (8.7)$$

$$[\, \boldsymbol{J_0}, \boldsymbol{J_1}\,] = \boldsymbol{J_1}\, .$$

They are proved by inserting (8.6) and comparing with (8.4). Making use of (8.6) and (8.7) the operator $\boldsymbol{J}^{\boldsymbol{2}}$ can be written this way

$$\boldsymbol{J}^2 = \boldsymbol{J_x}^2 + \boldsymbol{J_y}^2 + \boldsymbol{J_z}^2 = -\boldsymbol{J_1}\,\boldsymbol{J_{-1}} - \boldsymbol{J_{-1}}\,\boldsymbol{J_1} + \boldsymbol{J_0}^2 \equiv \sum_\mu (-1)^\mu \boldsymbol{J_\mu J_{-\mu}} = \boldsymbol{J_0}^2 - \boldsymbol{J_0} - 2\boldsymbol{J_1 J_{-1}}\, . \qquad (8.8)$$

8.2 The angular momentum operator expressed in terms of *d*-boson operators

We will construct operators in terms of creation and annihilation operators of *d*-bosons which satisfy the conditions (8.7) and which we name $\boldsymbol{J_1}$, $\boldsymbol{J_{-1}}$ and $\boldsymbol{J_0}$. At the same time, making use of (8.6) one obtains operators $\boldsymbol{J_x}$, $\boldsymbol{J_y}$ and $\boldsymbol{J_z}$, which fulfil the rules (8.4). The operator $\boldsymbol{J}^2$ is constructed, which satisfies an eigenvalue equation simultaneously with $\boldsymbol{J_z}$ (8.5).

In the IBM1 the following expressions for the “ladder“ operators are used

$$\boldsymbol{J}_{\boldsymbol{\mu}} = \sqrt{10}\, [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}_{\mu}\, , \;\; \mu = 1, 0, -1. \qquad (8.9)$$

With the help of (A8.4), one recognizes that these operators satisfy the commutation relations (8.7). In order to show that, we put $J' = J'' = 1$ in the equation (A8.4) which yields $k = 1$. We obtain

$$10[[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}_{M'}\, , [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}_{M''}] =$$

$$10[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}_{M'+M''}\, 3(1\; M'\, 1\; M'' \mid 1, M' + M'')\{^1{}_2\; ^1{}_2\; ^1{}_2\}(-2).$$

Because of $\{^1{}_2\; ^1{}_2\; ^1{}_2\} = 1/(6\sqrt{5})$ (see A3.13), (8.9) and

$(1, -1\; 1\; 0 \mid 1, -1) = (1, -1\; 1\; 1 \mid 1\; 0) = (1\; 0\; 1\; 1 \mid 1\; 1) = -1/\sqrt{2}$ we have

$$10[[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}_{M'}\, , [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}_{M''}] = \sqrt{10}[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}_{M'+M''} \;\text{ with } M' < M'' \qquad (8.10)$$

in agreement with (8.7).

As an example we check the effect of the operators $\boldsymbol{J_0}$ and $\boldsymbol{J}^2$ on the single-boson state $\boldsymbol{d}^+{}_\nu \,|\rangle$.

$$\boldsymbol{J_0}\, \boldsymbol{d}^+{}_\nu\, |\rangle = \sqrt{10}\, [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(1)}{}_0\, \boldsymbol{d}^+{}_\nu\, |\rangle =$$

$$\sqrt{10}\, \sum_\mu (2\; \mu\; 2, -\mu \mid 1\; 0) \boldsymbol{d}^+{}_\mu (-1)^{-\mu}\, \boldsymbol{d}_\mu \boldsymbol{d}^+{}_\nu |\rangle. \qquad (8.11)$$

Because of $\boldsymbol{d}_\mu \boldsymbol{d}^+{}_\nu = \delta_{\mu\nu} + \boldsymbol{d}^+{}_\nu\, \boldsymbol{d}_\mu$ and $(2\; \nu\; 2, -\nu \mid 1\; 0) = (-1)^{-\nu} \nu /\sqrt{10}$ (A1.16) the relation

$$\boldsymbol{J_0}\, \boldsymbol{d}^+{}_\nu\, |\rangle = \sqrt{10}\, (2\; \nu\; 2, -\nu \mid 1\; 0)(-1)^{-\nu} \boldsymbol{d}^+{}_\nu |\rangle = \nu\, \boldsymbol{d}^+{}_\nu |\rangle \qquad (8.12)$$

follows in agreement with (8.5). Furthermore, with the aid of (8.8) we calculate

$$\boldsymbol{J}^2\boldsymbol{d}^+_\nu\,|\rangle = (\boldsymbol{J_0}^2 - \boldsymbol{J_0} - 2\boldsymbol{J_1}\boldsymbol{J_{-1}})\,\boldsymbol{d}^+_\nu\,|\rangle =$$

$$(\nu^2 - \nu - 2{\cdot}10\,[\boldsymbol{d}^+\times\boldsymbol{d}^\sim]^{(1)}{}_1[\boldsymbol{d}^+\times\boldsymbol{d}^\sim]^{(1)}{}_{-1})\,\boldsymbol{d}^+_\nu\,|\rangle.$$

Due to $(2,\,\nu - 1,\,2,\,-\nu\ |\ 1,\,-1)^2 = (3-\nu)(2+\nu)/20$ (A1.16) we obtain

$2{\cdot}10\,[\boldsymbol{d}^+\times\boldsymbol{d}^\sim]^{(1)}{}_1[\boldsymbol{d}^+\times\boldsymbol{d}^\sim]^{(1)}{}_{-1}\,\boldsymbol{d}^+_\nu\,|\rangle =$

$2{\cdot}10\,\sum_{\lambda_1\lambda_2}\sum_{\mu_1\mu_2}(2\,\lambda_1\,2\,\lambda_2\,|\,1\,1)(2\,\mu_1\,2\,\mu_2\,|\,1,\,-1)\,\cdot$

$\boldsymbol{d}^+_{\lambda_1}(-1)^{\lambda_2}\boldsymbol{d}_{-\lambda_2}\boldsymbol{d}^+_{\mu_1}(-1)^{\mu_2}\boldsymbol{d}_{-\mu_2}\boldsymbol{d}^+_\nu\,|\rangle =$

$2{\cdot}10\,\sum_{\lambda_1\lambda_2}(2\,\lambda_1\,2\,\lambda_2\,|\,1\,1)(2,\,\nu-1,\,2,\,-\nu\,|\,1,\,-1)\boldsymbol{d}^+_{\lambda_1}(-1)^{\lambda_2}\boldsymbol{d}_{-\lambda_2}\,\boldsymbol{d}^+_{\nu-1}(-1)^{-\nu}\,|\rangle =$

$-2{\cdot}10(2,\,\nu-1,\,2,\,-\nu\ |\ 1,\,-1)^2\,\boldsymbol{d}^+_\nu\,|\rangle = -(3-\nu)(2+\nu)\boldsymbol{d}^+_\nu\,|\rangle$.

That is $\boldsymbol{J}^2\boldsymbol{d}^+_\nu\,|\rangle = (\nu^2 - \nu + 6 + \nu - \nu^2)\,^2\boldsymbol{d}^+_\nu\,|\rangle = 2(2+1)\boldsymbol{d}^+_\nu\,|\rangle$. (8.13)

The result (8.13) agrees with (8.5). According to (A1.22) and (A1.26) $\boldsymbol{J_0}$ and $\boldsymbol{J}^{\mathbf{2}}$ act analogously on every coupled state like $[\boldsymbol{d}^+\times\boldsymbol{d}^+]^{(J)}{}_M\,|\rangle$ which can be generalised to all states of the spherical basis (3.8, 6.18).

With the help of (8.8) and (8.9) we write the operators $\boldsymbol{J_z}$ and $\boldsymbol{J}^2$ explicitly making use of (6.19), (8.11) and (A1.16) as follows

$\boldsymbol{J_z} = \boldsymbol{J_0} = \sqrt{10}\,[\boldsymbol{d}^+\times\boldsymbol{d}^\sim]^{(1)}{}_0 = \sqrt{10}\,\sum_\mu(2\,\mu\,2,\,-\mu\,|\,1\,0)\boldsymbol{d}^+_\mu\boldsymbol{d}^\sim_{-\mu} = \sum_\mu\mu\,\boldsymbol{d}^+_\mu\boldsymbol{d}_\mu$,

$\boldsymbol{J}^2 = \sum_\mu(-1)^\mu\,\boldsymbol{J_\mu}\boldsymbol{J}_{-\mu} = -\sqrt{3}\sum_\mu(-1)^{1-\mu}/\sqrt{3}\,\boldsymbol{J_\mu}\boldsymbol{J}_{-\mu} =$ (8.14)

$-\sqrt{3}\sum_\mu(1\,\mu\,1,\,-\mu\,|\,0\,0)\,\boldsymbol{J_\mu}\boldsymbol{J}_{-\mu} = -\sqrt{3}[\boldsymbol{J}\times\boldsymbol{J}]^{(0)} = -10\sqrt{3}{\cdot}[[\boldsymbol{d}^+\times\boldsymbol{d}^\sim]^{(1)}\times[\boldsymbol{d}^+\times\boldsymbol{d}^\sim]^{(1)}]^{(0)}$.

8.3 The conservation of angular momentum

At the end of the last chapter, we made a supposition which can be written as

$[\,\boldsymbol{J_z}\,,\,\boldsymbol{H}\,] = 0$ and $[\,\boldsymbol{J}^2\,,\,\boldsymbol{H}\,] = 0.$ (8.15)

We check the first commutator in (8.15) and pick just the most complex part of $\boldsymbol{H}$ (7.31) out, namely the terms with the coefficients c_J. They contain linear combinations of expressions like

$$\boldsymbol{d}^+_\mu\boldsymbol{d}^+_{\mu'}\boldsymbol{d_\nu}\boldsymbol{d_{\nu'}} \qquad (8.16)$$

with the condition $\mu + \mu' - \nu - \nu' = 0$ (8.17)

which results from (6.19) and from the coupling yielding the angular momentum zero. We let $\boldsymbol{J_z}$ act on a term like (8.16) and push it step by step to the right hand side as follows

$$\boldsymbol{J_z}\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}} = \sum_{\mu 0} \mu_0\, \boldsymbol{d}^+_{\mu 0}\, \boldsymbol{d}_{\mu 0}\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}} =$$

$$\sum_{\mu 0} \mu_0\, (\boldsymbol{d}^+_{\mu 0}\, \boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}}\, \delta_{\mu 0\, \mu} + \boldsymbol{d}^+_{\mu}\, \boldsymbol{d}^+_{\mu 0}\, \boldsymbol{d}_{\mu 0}\, \boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}})\,, \qquad (8.18)$$

where the commutation relations

$$\boldsymbol{d_{\mu 0}}\, \boldsymbol{d}^+_{\mu} = \delta_{\mu 0\, \mu} + \boldsymbol{d}^+_{\mu}\, \boldsymbol{d_{\mu 0}} \qquad (8.19)$$

have been applied. The first sum on the right hand side of (8.18) yields one single term. An analogous step results in

$$\boldsymbol{J_z}\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}} = \qquad (8.20)$$

$$\mu\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}} + \sum_{\mu 0} \mu_0\, (\boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu 0}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}}\, \delta_{\mu 0\, \mu'} + \boldsymbol{d}^+_{\mu}\, \boldsymbol{d}^+_{\mu'}\, \boldsymbol{d}^+_{\mu 0}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}}\, \boldsymbol{d}_{\mu 0}\,).$$

The following step produces

$$\boldsymbol{J_z}\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}} = (\mu + \mu' - \nu)\;\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}} + \sum_{\mu 0} \mu_0\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\boldsymbol{d_\nu}\, \boldsymbol{d}^+_{\mu 0}\, \boldsymbol{d_{\nu'}}\, \boldsymbol{d}_{\mu 0} =$$

$$(\mu + \mu' - \nu\, - \nu')\;\, \boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}} + \boldsymbol{d}^+_{\mu}\, \boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}}\, \boldsymbol{J_z}. \qquad (8.21)$$

The last term but one in (8.21) vanishes because of (8.17). Therefore $\boldsymbol{J_z}$ commutes with $\boldsymbol{d}^+_{\mu}\boldsymbol{d}^+_{\mu'}\, \boldsymbol{d_\nu}\, \boldsymbol{d_{\nu'}}$ and consequently with the first sum in (7.31). In the same way one shows that $\boldsymbol{J_z}$ commutes with expressions such as

$$\boldsymbol{d}^+_{\mu}\, \boldsymbol{d_\mu}\,,\; \boldsymbol{d}^+_{\mu}\, \boldsymbol{d}^+_{-\mu}\,,\; \boldsymbol{d_\mu}\, \boldsymbol{d}_{-\mu}\,,\; \boldsymbol{d}^+_{\mu}\, \boldsymbol{d}^+_{\mu'}\, \boldsymbol{d}_{\nu}\;\; (\mu + \mu' - \nu = 0) \text{ etc.}$$

Therefore [$\boldsymbol{J_z}$, $\boldsymbol{H}$] = 0 (8.15) is satisfied.

Since in $\boldsymbol{H}$ no spatial axis prevails the analogous relations [$\boldsymbol{J_x}$, $\boldsymbol{H}$] = [$\boldsymbol{J_y}$, $\boldsymbol{H}$] = 0 and consequently [$\boldsymbol{J}^2$, $\boldsymbol{H}$] = 0 (see 8.15) are true. From this, the well-known fact follows that the eigenstates of $\boldsymbol{H}$ have a defined angular momentum J.

9 The Hamiltonian expressed in terms of Casimir operators

In this chapter the Hamilton operator (7.17) will be transformed to a linear combination of Casimir operators. These are characteristic of Lie algebras (chapter 13) and permit to express the eigenenergies of the IBM in a closed form in some special cases. Essentially, we follow the reports of Castaños et al. (1979) and Eisenberg and Greiner (1987).

In chapter 14 we will show that the number operator $\boldsymbol{n_d}$ of d-bosons (6.35), the number operator $\boldsymbol{N}$ of all bosons (6.37) and the angular momentum operator $\boldsymbol{J}^2$ belong to the type of Casimir operators. We give them again

$$\boldsymbol{n_d} = \sqrt{5}\,[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)} = \sum_\mu \boldsymbol{d}^+_\mu \boldsymbol{d}_\mu,$$

$$\boldsymbol{N} = \boldsymbol{s}^+\boldsymbol{s} + \boldsymbol{n_d}, \tag{9.1}$$

$$\boldsymbol{J}^2 = -\sqrt{3}\cdot 10\,[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}]^{(0)}.$$

The operator

$$\boldsymbol{T}^2 = \boldsymbol{n_d}(\boldsymbol{n_d} + 3) - 5\cdot[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}\,[\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)} \tag{9.2}$$

is also of this kind and we name it seniority operator. It shows a certain relationship to the following operator

$$\boldsymbol{R}^2 = \boldsymbol{N}(\boldsymbol{N} + 4) - (\sqrt{5}\cdot[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} - \boldsymbol{s}^+\boldsymbol{s}^+)(\sqrt{5}[\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)} - \boldsymbol{ss}). \tag{9.3}$$

$\boldsymbol{R}^2$ is identical with $\boldsymbol{C}_{o(6)}$ (14.53) but it differs from the corresponding expression employed by Eisenberg and Greiner (1987, p.398) in the signs of $\boldsymbol{s}^+\boldsymbol{s}^+$ and $\boldsymbol{ss}$. The operator

$$\boldsymbol{Q}^2 = \sqrt{5}\cdot[\boldsymbol{Q} \times \boldsymbol{Q}]^{(0)} = \sum_\mu (-1)^\mu\, \boldsymbol{Q}_\mu \boldsymbol{Q}_{-\mu}$$

with $$\boldsymbol{Q}_\mu = \boldsymbol{d}^+_\mu \boldsymbol{s} + \boldsymbol{s}^+\boldsymbol{d}^\sim_\mu - (\sqrt{7}/2)\,[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu \tag{9.4}$$

is related to the quadruple moment of the nucleus. Several authors place a positive sign in front of the last term in (9.4).

Now we replace all components of the Hamilton operator (7.17) step by step by expressions constituted by the operators (9.1) up to (9.4). First, we deal with terms containing exclusively d-operators. Besides $\boldsymbol{n_d}$ in (7.17) we have

$$\tfrac{1}{2}\, c_J \sqrt{(2J + 1)}\,[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)}, \qquad J = 0, 2, 4. \tag{9.5}$$

It is easy to treat the term with $J = 0$ in (9.5). The relation (9.2) reveals

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)}]^{(0)} = (1/5)(\boldsymbol{n_d}(\boldsymbol{n_d} + 3) - \boldsymbol{T}^2). \tag{9.6}$$

Since the operators $\boldsymbol{n_d}$, $\boldsymbol{J}^2$ and $\boldsymbol{Q_\mu}$ are built with terms of the type $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}{}_M$, representing the operators $[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)}$ by the same terms is advisable. According to (A4.2) and (A4.3) and analogously to (7.26) the following relation holds

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} = \sum_{J'} (-1)^{J'} \sqrt{(2J+1)}\sqrt{(2J'+1)}\{^2{}_2\,{}^2{}_2\,{}^{J'}{}_J\} \cdot$$

$$\sum_{Mm\,\mu} (J'\,M\,J', -M \mid 0\;0)(2\;m\;2, M-m \mid J'\,M)(2\;\mu\;2, -M-\mu \mid J', -M)\cdot$$

$$\boldsymbol{d}^+{}_m\,\boldsymbol{d}^+{}_\mu\,\boldsymbol{d}^\sim{}_{M-m}\,\boldsymbol{d}^\sim{}_{-M-\mu}\,. \tag{9.7}$$

Almost with the same method which yielded the equations (7.29) and (7.30), starting from (9.7) one obtains

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} =$$

$$\sum_{J'} (-1)^{J'} \sqrt{(2J+1)}\sqrt{(2J'+1)}\{^2{}_2\,{}^2{}_2\,{}^{J'}{}_J\} \cdot [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)} -$$

$$(1/5)\sqrt{(2J+1)}\boldsymbol{n_d}\,. \tag{9.8}$$

Consequently, we now treat the terms

$$\boldsymbol{W_{J'}} \equiv [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)} \tag{9.9}$$

in order to represent the expressions (9.5) showing J = 2 and 4 by means of Casimir operators. First, we take from (9.1) the following relations

$$\boldsymbol{W_0} \equiv [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}]^{(0)} = (1/5)\boldsymbol{n_d}^2 \tag{9.10}$$

and $$\boldsymbol{W_1} \equiv [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}]^{(0)} = -(10\sqrt{3})^{-1}\boldsymbol{J}^2. \tag{9.11}$$

For the operators $\boldsymbol{W_2}$, $\boldsymbol{W_3}$ and $\boldsymbol{W_4}$ we draw up a system of linear equations. For that purpose we represent the left-hand side of (9.6) with (9.8) too and get by means of (A3.12)

$$\boldsymbol{n_d}(\boldsymbol{n_d} + 3) - \boldsymbol{T}^2 + \boldsymbol{n_d} = \sum_{J'} \sqrt{(2J'+1)} \cdot [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)}\,. \tag{9.12}$$

We obtain further equations from the fact that the terms $\boldsymbol{W_J}$ are linearly dependent on one another owing to their symmetry. In order to make this visible, first we interchange the d-operators in $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}{}_M$:

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}{}_M = \sum_\mu (2\;\mu\;2, M-\mu \mid J\,M)(-1)^{M-\mu}\boldsymbol{d}^+{}_\mu\,\boldsymbol{d}_{\mu-M} = \tag{9.13}$$

$$-\sum_\mu (2\;\mu\;2, M-\mu \mid J\,M)(-1)^{M-\mu}\,\delta_{\mu,\,\mu-M} + (-1)^J[\boldsymbol{d}^\sim \times \boldsymbol{d}^+]^{(J)}{}_M\,.$$

In particular the expression

$$\sum_\mu (2\;\mu\;2, M-\mu \mid J\,M)(-1)^{M-\mu}\,\delta_{\mu,\,\mu-M} =$$

$$\sum_\mu (2\;\mu\;2, -\mu \mid J\;0)(-1)^{-\mu} = \sqrt{5}\sum_\mu (2\;\mu\;2, -\mu \mid J\;0)(-1)^{-\mu}/\sqrt{5} = \tag{9.14}$$

$$\sqrt{5}\sum_\mu (2\;\mu\;2, -\mu \mid J\;0)(2\;\mu\;2, -\mu \mid 0\;0) = \sqrt{5}\cdot\delta_{J0}$$

can be inserted in (9.13) yielding the following equation

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}{}_M = -\sqrt{5}\cdot\delta_{J0} + (-1)^J[\boldsymbol{d}^\sim \times \boldsymbol{d}^+]^{(J)}{}_M. \qquad (9.15)$$

The inverse relation reads $[\boldsymbol{d}^\sim \times \boldsymbol{d}^+]^{(J)}{}_M = \sqrt{5}\cdot\delta_{J0} + (-1)^J[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}{}_M$. (9.16)

Bringing (9.15) in the expression (9.9) we obtain

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)} = (-1)^{J'} [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^+]^{(J')}]^{(0)} - \sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}\,\delta_{J'0}. \qquad (9.17)$$

Now we interchange both facing operators $\boldsymbol{d}^\sim$ on the right hand side of (9.17) by means of (A4.1) and (A4.2) which results in

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)} = -\sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}\,\delta_{J',0} + \qquad (9.18)$$

$$(-1)^{J'} \sum_{J''} \sqrt{(2J'+1)}\sqrt{(2J''+1)}\{^{2}{}_{2}\,{}^{2}{}_{2}\,{}^{J'}{}_{J''}\}(-1)^{J'+J''}[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^+]^{(J'')}]^{(0)}.$$

In (9.18) we interchange the boson operators of the expression $[\boldsymbol{d}^\sim \times \boldsymbol{d}^+]^{(J'')}$ (9.16) and obtain

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)} = -\sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}\,\delta_{J',0} +$$

$$\sqrt{(2J'+1)}\sum_{J''}\sqrt{(2J''+1)}\{^{2}{}_{2}\,{}^{2}{}_{2}\,{}^{J'}{}_{J''}\}[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')}]^{(0)} +$$

$$\sqrt{(2J'+1)}\sum_{J''}\sqrt{(2J''+1)}\{^{2}{}_{2}\,{}^{2}{}_{2}\,{}^{J'}{}_{J''}\}\sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}\,\delta_{J'',0} =$$

$$-\sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}\,\delta_{J',0} +$$

$$\sqrt{(2J'+1)}\sum_{J''}\sqrt{(2J''+1)}\{^{2}{}_{2}\,{}^{2}{}_{2}\,{}^{J'}{}_{J''}\}[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')}]^{(0)} +$$

$$\sqrt{(2J'+1)}(-1)^{J'}(1/\sqrt{5})[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}. \qquad (9.19)$$

In the last term we made use of (A3.12). Employing (A3.12) we write (9.19) for $J' = 0$ and 1 in the following form

$$\boldsymbol{W_0} \equiv [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}]^{(0)} =$$

$$(1/5)\sum_{J''}(-1)^{J''}\sqrt{(2J''+1)}[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')}]^{(0)} +$$

$$(1/\sqrt{5})[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)} - \sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}, \qquad (9.20)$$

$$\boldsymbol{W_1} \equiv [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}]^{(0)} =$$

$$\sqrt{3}\sum_{J''}\sqrt{(2J''+1)}\{^{2}{}_{2}\,{}^{2}{}_{2}\,{}^{1}{}_{J''}\}[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')}]^{(0)} - \sqrt{(3/5)}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}. \qquad (9.21)$$

We calculate the 6-*j* symbols with the aid of (A3.13) and insert (9.9) up to (9.11) in (9.12), (9.20) and (9.21). Doing so and making use of (A3.13), we obtain an inhomogeneous and linear system of equations for $\boldsymbol{W_2}$, $\boldsymbol{W_3}$ and $\boldsymbol{W_4}$ (9.9). We write down the solutions of the system and add the expressions (9.10) and (9.11) as follows

$$\boldsymbol{W_0} = (1/5)\boldsymbol{n_d}^2,$$

$$\boldsymbol{W_1} = -(10\sqrt{3})^{-1}\boldsymbol{J}^2,$$

$$\boldsymbol{W_2} = 2(7\sqrt{5})^{-1}(\boldsymbol{n_d}^2 + 5\boldsymbol{n_d} - \boldsymbol{T}^2 + (1/4)\boldsymbol{J}^2), \tag{9.22}$$

$$\boldsymbol{W_3} = -(2\sqrt{7})^{-1}(\boldsymbol{T}^2 - (1/5)\boldsymbol{J}^2),$$

$$\boldsymbol{W_4} = (1/7)((6/5)\boldsymbol{n_d}^2 + 6\boldsymbol{n_d} - (1/2)\boldsymbol{T}^2 - (1/6)\boldsymbol{J}^2).$$

Inserting these expressions in (9.8) yields for $J = 2$

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} = (7\sqrt{5})^{-1}(2\boldsymbol{n_d}^2 - 4\boldsymbol{n_d} + 2\boldsymbol{T}^2 - \boldsymbol{J}^2). \tag{9.23}$$

In order to calculate the corresponding expression for $J = 4$ we take the 6-*j* symbols from (A3.14) and obtain

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(4)} \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(4)}]^{(0)} =$$

$$(1/7)((6/5)\boldsymbol{n_d}^2 - (12/5)\boldsymbol{n_d} - (1/5)\boldsymbol{T}^2 + (1/3)\boldsymbol{J}^2). \tag{9.24}$$

Thus we have treated all terms in the Hamiltonian (7.17) which are constituted exclusively by *d*-operators and we are turning to the other terms. Owing to (7.22) and (7.23) we have

$$[\boldsymbol{s}^+\boldsymbol{s}^+ \times \boldsymbol{ss}]^{(0)} = (\boldsymbol{N} - \boldsymbol{n}_{\mathrm{d}})(\boldsymbol{N} - \boldsymbol{n_d} - 1), \tag{9.25}$$

$$[\boldsymbol{d}^+\boldsymbol{s}^+ \times \boldsymbol{d}^{\sim}\boldsymbol{s}]^{(0)} = (1/\sqrt{5})(\boldsymbol{N} - \boldsymbol{n_d})\boldsymbol{n_d}\,. \tag{9.26}$$

The preceding term in (7.17) reads

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} \times \boldsymbol{ss}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{s}^+ \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(0)}]^{(0)} =$$

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}\boldsymbol{ss} + \boldsymbol{s}^+\boldsymbol{s}^+[\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(0)} =$$

$$-(1/\sqrt{5})(-\boldsymbol{R}^2 + \boldsymbol{N}(\boldsymbol{N} + 4) + \boldsymbol{T}^2 - \boldsymbol{n_d}(\boldsymbol{n_d} + 3) - \boldsymbol{s}^+\boldsymbol{s}^+\boldsymbol{ss}) = \tag{9.27}$$

$$-(1/\sqrt{5})(-\boldsymbol{R}^2 + \boldsymbol{T}^2 + \boldsymbol{N} + (\boldsymbol{N} - \boldsymbol{n_d})(2\boldsymbol{n_d} + 4))$$

where (9.2), (9.3) and (9.25) have been employed. In order to work on the remaining term in $\boldsymbol{H}$ (7.17) we write (9.4) this way

$$\boldsymbol{Q}^2/\sqrt{5} = [(\boldsymbol{d}^+\boldsymbol{s} + \boldsymbol{s}^+\boldsymbol{d}^{\sim} - (\sqrt{7}/2)[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)}) \times (\boldsymbol{d}^+\boldsymbol{s} + \boldsymbol{s}^+\boldsymbol{d}^{\sim} - (\sqrt{7}/2)[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)})]^{(0)}$$

which we transform to

$$\boldsymbol{Q}^2/\sqrt{5} = [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}\boldsymbol{ss} + \boldsymbol{s}^+\boldsymbol{s}^+[\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(0)} + (7/4)[[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} +$$

$$\boldsymbol{ss}^+[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(0)} + [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^+]^{(0)}\boldsymbol{s}^+\boldsymbol{s} -$$

$$(\sqrt{7}/2)\boldsymbol{s}([\boldsymbol{d}^+ \times [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} + [[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)} \times \boldsymbol{d}^+]^{(0)}) - \tag{9.28}$$

$$(\sqrt{7}/2)\boldsymbol{s}^+([\boldsymbol{d}^{\sim} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} + [[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)} \times \boldsymbol{d}^{\sim}]^{(0)}).$$

The first line in (9.28) contains the relation (9.27). We write the following term by means of (9.22) as

$$(7/4)[[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} = (2\sqrt{5})^{-1}(\boldsymbol{n_d}^{2} + 5\boldsymbol{n_d} - \boldsymbol{T}^{2} + (1/4)\boldsymbol{J}^{2}). \qquad (9.29)$$

The second line in (9.28) is represented with the help of (9.16) and of the relation $\boldsymbol{ss}^{+} = 1 + \boldsymbol{s}^{+}\boldsymbol{s}$ like this

$$\begin{aligned}
&\boldsymbol{ss}^{+}[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(0)} + [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{+}]^{(0)}\boldsymbol{s}^{+}\boldsymbol{s} = \\
&[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(0)} + [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(0)}\, \boldsymbol{s}^{+}\boldsymbol{s} + \sqrt{5}\, \boldsymbol{s}^{+}\boldsymbol{s} + [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(0)}\, \boldsymbol{s}^{+}\boldsymbol{s} = \\
&(1/\sqrt{5})\boldsymbol{n_d} + \sqrt{5}(\boldsymbol{N} - \boldsymbol{n_d}) + (2/\sqrt{5})\boldsymbol{n_d}(\boldsymbol{N} - \boldsymbol{n_d}) = \qquad (9.30) \\
&(1/\sqrt{5})(\boldsymbol{n_d} + (5 + 2\boldsymbol{n_d})(\boldsymbol{N} - \boldsymbol{n}_{\mathrm{d}})).
\end{aligned}$$

The third line in (9.28) contains the expression

$$[[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \times \boldsymbol{d}^{+}]^{(0)} = \sum_{\mu_2\, \mu'\, \mu_0} (2\ \mu_2\ 2\ \mu' \mid 2\ \mu_0)(2\ \mu_0\ 2, -\mu_0 \mid 0\ 0)\ \boldsymbol{d}^{+}_{\mu_2}\, \boldsymbol{d}^{\sim}_{\mu'}\boldsymbol{d}^{+}_{-\mu_0}.$$

With the aid of $\boldsymbol{d}^{+}_{\mu_2}\ \boldsymbol{d}^{\sim}_{\mu'}\ \boldsymbol{d}^{+}_{-\mu_0} = (-1)^{\mu'}\ (\boldsymbol{d}^{+}_{\mu_2}\ \delta_{\mu'\ \mu_0} + \boldsymbol{d}^{+}_{-\mu_0}\ \boldsymbol{d}^{+}_{\mu_2}\ \boldsymbol{d}_{-\mu'})$, $(2\ \mu_0\ 2, -\mu_0 \mid 0\ 0) = (-1)^{-\mu_0}/\sqrt{5}$, $(2\ \mu_2\ 2\ \mu_0 \mid 2\ \mu_0) = \delta_{\mu_2\ 0}\ (\mu_0^{2} - 2)/\sqrt{14}$ and of (A1.16) we obtain

$$\begin{aligned}
&[[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \times \boldsymbol{d}^{+}]^{(0)} = \\
&\underbrace{\sum_{\mu_0 = -2}^{2} \sum_{\mu_2} (2\ \mu_2\ 2\ \mu_0 \mid 2\ \mu_0)(1/\sqrt{5})\boldsymbol{d}^{+}_{\mu_2}}_{0} + [\boldsymbol{d}^{+} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} = \\
&\qquad\qquad 0 \qquad\qquad + [\boldsymbol{d}^{+} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} \qquad (9.31)
\end{aligned}$$

and recouple according to (A3.5) this way

$$\begin{aligned}
&[[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \times \boldsymbol{d}^{+}]^{(0)} = [\boldsymbol{d}^{+} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} = \qquad (9.32) \\
&\sum_{J_2} \sqrt{5}\ \sqrt{(2J_2 + 1)}\begin{Bmatrix} 2 & 2 & 2 \\ 2 & 0 & J_2 \end{Bmatrix}[[\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(J_2)}\boldsymbol{d}^{\sim}]^{(0)} = [[\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(2)} \times \boldsymbol{d}^{\sim}]^{(0)}.
\end{aligned}$$

The value of the 6-j symbol for $J_2 = 2$ has been taken from (A3.12). Analogously the relation

$$[\boldsymbol{d}^{\sim} \times [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} = [[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \times \boldsymbol{d}^{\sim}]^{(0)} = [\boldsymbol{d}^{+} \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)} \qquad (9.33)$$

holds so that the third and the fourth line in (9.28) can be written as follows

$$\begin{aligned}
&-(\sqrt{7}/2)(2\boldsymbol{s}[[\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(2)} \times \boldsymbol{d}^{\sim}]^{(0)} + 2\boldsymbol{s}^{+}[\boldsymbol{d}^{+} \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)}) = \\
&-\sqrt{7}([[\boldsymbol{d}^{+} \times \boldsymbol{d}^{+}]^{(2)} \times \boldsymbol{d}^{\sim}\boldsymbol{s}]^{(0)} + [\boldsymbol{s}^{+}\boldsymbol{d}^{+} \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)}). \qquad (9.34)
\end{aligned}$$

Apart from a factor, the expression (9.34) is just the remaining term in $\boldsymbol{H}$ (7.17). By means of (9.27) up to (9.30) and (9.34), we can write it this way

$$-\sqrt{(½)}\, v_2\,([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times \boldsymbol{d}^{\sim}\boldsymbol{s}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{d}^+ \times [\boldsymbol{d}^{\sim} \times \boldsymbol{d}^{\sim}]^{(2)}]^{(0)}) =$$

$$(1/\sqrt{14})\, v_2\,((1/\sqrt{5})\boldsymbol{Q}^2 + (1/\sqrt{5})(-\boldsymbol{R}^2 + \boldsymbol{T}^2 + \boldsymbol{N} + (\boldsymbol{N} - \boldsymbol{n_d})(2\boldsymbol{n_d} + 4)) -$$
$$(1/\sqrt{20})(\boldsymbol{n_d}^2 + 5\boldsymbol{n_d} - \boldsymbol{T}^2 + (1/4)\boldsymbol{J}^2) -$$
$$(1/\sqrt{5})(\boldsymbol{n_d} + (5 + 2\boldsymbol{n_d})(\boldsymbol{N} - \boldsymbol{n_d}))) = \qquad (9.35)$$
$$v_2\,(1/\sqrt{70})(\boldsymbol{Q}^2 + (3/2)\boldsymbol{T}^2 - \boldsymbol{R}^2 - (1/8)\boldsymbol{J}^2 - (1/2)\boldsymbol{n_d}^2 - (5/2)\,\boldsymbol{n_d}).$$

Now we are able to represent the Hamilton operator (7.17) by the operators cited in (9.1) up to (9.4) making use of (6.35), (6.37), (9.6), (9.23) up to (9.25), (9.27) and (9.35) like this

$$\boldsymbol{H} = \varepsilon_s \boldsymbol{N} + (\varepsilon_d - \varepsilon_s)\boldsymbol{n_d} + c_0\,(1/10)(-\boldsymbol{T}^2 + \boldsymbol{n_d}\,(\boldsymbol{n_d} + 3)) +$$
$$c_2\,(1/14)(2\boldsymbol{n_d}^2 + 2\boldsymbol{T}^2 - 4\boldsymbol{n_d} - \boldsymbol{J}^2) +$$
$$c_4(3/14)((6/5)\boldsymbol{n_d}^2 - (12/5)\boldsymbol{n_d} - (1/5)\boldsymbol{T}^2 + (1/3)\boldsymbol{J}^2) -$$
$$v_2\,(1/\sqrt{70})(\boldsymbol{Q}^2 + (3/2)\boldsymbol{T}^2 - \boldsymbol{R}^2 - (1/8)\boldsymbol{J}^2 - (1/2)\boldsymbol{n_d}^2 - (5/2)\boldsymbol{n_d}) -$$
$$v_o\,(1/\sqrt{20})(-\boldsymbol{R}^2 + \boldsymbol{T}^2 + 5\boldsymbol{N} + 2\boldsymbol{N}\boldsymbol{n_d} - 2\boldsymbol{n_d}^2 - 4\boldsymbol{n_d}) + \qquad (9.36)$$
$$u_2\,(1/\sqrt{5})(\boldsymbol{N}\boldsymbol{n}_{\mathrm{d}} - \boldsymbol{n_d}^2) + u_0(1/2)(\boldsymbol{N}^2 - 2\boldsymbol{N}\boldsymbol{n_d} - \boldsymbol{N} + \boldsymbol{n_d}^2 + \boldsymbol{n_d}).$$

We summarise

$$\boldsymbol{H} = \varepsilon_n \boldsymbol{N} + \varepsilon_d'\, \boldsymbol{n}_d + v_n \boldsymbol{N}^2 + v_{n\,d}\, \boldsymbol{N}\boldsymbol{n_d} + v_d\, \boldsymbol{n_d}^2 + v_r \boldsymbol{R}^2 + v_t \boldsymbol{T}^2 + v_j \boldsymbol{J}^2 + v_q \boldsymbol{Q}^2 \qquad (9.37)$$

with

$$\varepsilon_n = \varepsilon_s - (\sqrt{5}/2)v_0 - (1/2)u_0\,,$$
$$\varepsilon_d' = \varepsilon_d - \varepsilon_s + (3/10)c_0 - (2/7)c_2 - (18/35)c_4 + (5/\sqrt{280})v_2 + (2/\sqrt{5})v_0 + (1/2)u_0\,,$$
$$v_n = (1/2)u_0\,, \qquad (9.38)$$
$$v_{nd} = -\,(1/\sqrt{5})v_0 + (1/\sqrt{5})u_2 - u_0\,,$$
$$v_d = (1/10)c_0 + (1/7)c_2 + (9/35)c_4 + \sqrt{(1/\sqrt{280})}v_2 + (1/\sqrt{5})v_0 - (1/\sqrt{5})u_2 + (1/2)u_0\,,$$
$$v_r = (1/\sqrt{70})v_2 + (1/\sqrt{20})v_0\,,$$
$$v_t = -\,(1/10)c_0 + (1/7)c_2 - (3/70)c_4 - (3/\sqrt{280})v_2 - (1/\sqrt{20})v_0\,,$$
$$v_j = -\,(1/14)c_2 + (1/14)c_4 + (1/\sqrt{4480})v_2,$$
$$v_q = -\,(1/\sqrt{70})v_2\,.$$

All summands with a coefficient v_2 change their sign if $\boldsymbol{Q}_{\mu}$ is defined with a plus sign in front of the last term in (9.4).

In (9.37) the Hamilton operator of the IBM1 is represented by the Casimir operators (9.1) up to (9.4). This is the so-called multipole form. In the same way as in (7.21) we replace the operator $\boldsymbol{N}$ by its eigenvalue N and obtain

$$\boldsymbol{H} = \varepsilon_n N + v_n N^2 + (\varepsilon_d' + v_{nd} N)\boldsymbol{n_d} + v_d \boldsymbol{n_d}^2 + v_r \boldsymbol{R}^2 + v_t \boldsymbol{T}^2 + v_j \boldsymbol{J}^2 + v_q \boldsymbol{Q}^2. \quad (9.39)$$

If only a single nucleus is investigated the term $\varepsilon_n N + v_n N^2$ is omitted. Then the eigenstates depend solely on the following six parameters

$$\varepsilon_d'' = \varepsilon_d' + v_{nd} N\ ,\ v_d\ ,\ v_r\ ,\ v_t\ ,\ v_j \text{ and } v_q \quad (9.40)$$

analogously to (7.33).

In order to solve eigenvalue problems usually the Hamilton matrix is diagonalised (chapter 12). However, the interacting boson model has the peculiarity that for special cases (limits) the eigenenergies can be given analytically. These cases are found by allocating the value zero to several coefficients in (9.39). In chapters 10 and 14 we will deal with the physical significance of this measure and of the operators (9.1) up to (9.4) and look at the analytical solutions.

10 The *u*(5)- or vibrational limit

At the end of the last chapter the fact was mentioned briefly that the eigenvalues of the Hamilton operator $\boldsymbol{H}$ (9.39) can be written in a closed form if certain coefficients of this operator are zero. We deal here with a special case characterised by vanishing coefficients v_r and v_q , which is realised approximately by a group of real nuclei.

In the first section we will write down this special operator and remember the spherical basis of the boson model. The second section reveals that this basis contains the eigenfunctions of the seniority operator $\boldsymbol{T}^{\,2}$. In section 10.3 calculated nuclear spectra of the relevant special case will be compared with measured ones.

10.1 The Hamiltonian of the vibrational limit. The spherical basis

We will formulate now the first special case of the Hamiltonian by putting $v_r = v_q = 0$. This has the consequence that v_0 and v_2 vanish (see 9.38) and that in $\boldsymbol{H}$ (7.17) remain only the terms which conserve both the number of *d*-bosons and the one of the *s*-bosons ($\boldsymbol{d}^{\,+}$ is coupled to $\boldsymbol{d}^{\,\sim}$ and $\boldsymbol{s}^{+}$ is connected with $\boldsymbol{s}$). The Hamilton operator of this approximation reads according to (9.39)

$$\boldsymbol{H}^{(\mathrm{I})} = \varepsilon_n N + v_n N^2 + (\varepsilon_d{}' + v_{nd} N)\boldsymbol{n_d} + v_d\,\boldsymbol{n_d}^2 + v_t \boldsymbol{T}^2 + v_j \boldsymbol{J}^2 \qquad (10.1)$$

with
$$\varepsilon_n = \varepsilon_s - (1/2)u_0\,,$$
$$\varepsilon_d{}' = \varepsilon_d - \varepsilon_s + (3/10)c_0 - (2/7)c_2 - (18/35)c_4 + (1/2)u_0\,,$$
$$v_n = (1/2)u_0\,,$$
$$v_{nd} = (1/\surd 5)u_2 - u_0\,, \qquad (10.2)$$
$$v_d = (1/10)c_0 + (1/7)c_2 + (9/35)c_4 - (1/\surd 5)u_2 + (1/2)u_0\,,$$
$$v_t = -(1/10)c_0 + (1/7)c_2 - (3/70)c_4\,,$$
$$v_j = -(1/14)c_2 + (1/14)c_4\,.$$

The coefficients in (10.2) result from the equations (9.38) taking into account the condition $v_0 = v_2 = 0$. Because the operators in (10.1) act only on *d*-boson states, the eigenfunctions of $\boldsymbol{H}^{(\mathrm{I})}$ are products of pure *s*-boson states and *d*-boson states undergoing the symmetrisation procedure. The states of the spherical basis (3.8) and (6.18) have this structure and we will show in this chapter that they are really eigenfunctions of $\boldsymbol{H}^{(\mathrm{I})}$. They are characterised by the integers N (number of bosons), n_d (number of *d*-bosons), τ (seniority), n_Δ (number of *d*-boson triples resulting in the angular momentum zero) and J (angular momentum) and are written like this

$$|N\, n_d\, \tau\, n_\Delta\, J\,\rangle. \qquad (10.3)$$

In this chapter we drop the projection M of J. According to (6.33), (6.34) and to the section 8.2 the following eigenvalue equations hold

$$\boldsymbol{n_d}\,|N\, n_d\, \tau\, n_\Delta\, J\,\rangle = n_d\,|N\, n_d\, \tau\, n_\Delta\, J\,\rangle \tag{10.4}$$

$$\boldsymbol{J}^2\,|N\, n_d\, \tau\, n_\Delta\, J\,\rangle = J(J+1)\,|N\, n_d\, \tau\, n_\Delta\, J\,\rangle.$$

Owing to (3.8) and (3.9) for J only the following values are allowed

$$J = \lambda,\ \lambda + 1,\ \ldots\ , 2\lambda - 3,\ 2\lambda - 2,\ 2\lambda \quad \text{with } \lambda = \tau - 3n_\Delta \text{ and } \tau \le n_d. \tag{10.5}$$

10.2 Eigenvalues of the seniority operator

We now investigate the effect of the operator $\boldsymbol{T}^2$ (9.2) on a state of the spherical basis (10.3). By means of (9.2), (6.12) and (6.28) we write

$$\boldsymbol{T}^2 = \boldsymbol{n_d}(\boldsymbol{n_d} + 3) - \boldsymbol{P}^+\boldsymbol{P}^- \text{ with the following details} \tag{10.5a}$$

$$\boldsymbol{n_d} = \textstyle\sum_\mu \boldsymbol{d}^+_\mu \boldsymbol{d_\mu}, \quad \boldsymbol{P}^+ = \sum_\mu (-1)^\mu \boldsymbol{d}^+_\mu \boldsymbol{d}^+_{-\mu}, \quad \boldsymbol{P}^- = \sum_\mu (-1)^\mu \boldsymbol{d_\mu}\,\boldsymbol{d_{-\mu}}.$$

We look into several commutators of these operators. First we claim

$$[\boldsymbol{n_d}\,,\, \boldsymbol{P}^+] = 2\boldsymbol{P}^+, \tag{10.6}$$

which we prove with the help of (5.36)

$$[\,\textstyle\sum_\mu \boldsymbol{d}^+_\mu \boldsymbol{d_\mu}\,,\, 2\boldsymbol{d}^+_2 \boldsymbol{d}^+_{-2} - 2\boldsymbol{d}^+_1 \boldsymbol{d}^+_{-1} + \boldsymbol{d}^{+2}_0] = [\boldsymbol{d}^+_2 \boldsymbol{d}_2 + \boldsymbol{d}^+_{-2} \boldsymbol{d}_{-2}\,,\, 2\boldsymbol{d}^+_2 \boldsymbol{d}^+_{-2}] -$$

$$[\boldsymbol{d}^+_1 \boldsymbol{d}_1 + \boldsymbol{d}^+_{-1} \boldsymbol{d}_{-1}\,,\, 2\boldsymbol{d}^+_1 \boldsymbol{d}_{-1}] + [\boldsymbol{d}^+_0 \boldsymbol{d}_0\,,\, \boldsymbol{d}^{+2}_0] =$$

$$2(2\boldsymbol{d}^+_2 \boldsymbol{d}^+_{-2} - 2\boldsymbol{d}^+_1 \boldsymbol{d}^+_{-1} + \boldsymbol{d}^{+2}_0) = 2\boldsymbol{P}^+.$$

The validity of the relation

$$[\boldsymbol{P}^-\,,\, \boldsymbol{P}^+] = 2(2\boldsymbol{n_d} + 5) \tag{10.7}$$

is shown this way

$$[\,\textstyle\sum_\mu (-1)^\mu \boldsymbol{d}_\mu \boldsymbol{d}_{-\mu}\,,\, \sum_\mu (-1)^\mu \boldsymbol{d}^+_\mu \boldsymbol{d}^+_{-\mu}\,] =$$

$$4[\boldsymbol{d}_2 \boldsymbol{d}_{-2}\,,\, \boldsymbol{d}^+_2 \boldsymbol{d}^+_{-2}] + 4[\boldsymbol{d}_1 \boldsymbol{d}_{-1}\,,\, \boldsymbol{d}^+_1 \boldsymbol{d}^+_{-1}] + [\boldsymbol{d}_0^2\,, \boldsymbol{d}^{+2}_0] =$$

$$4(1 + \boldsymbol{d}^+_2 \boldsymbol{d}_2 + \boldsymbol{d}^+_{-2} \boldsymbol{d}_{-2}) + 4(1 + \boldsymbol{d}^+_1 \boldsymbol{d}_1 + \boldsymbol{d}^+_{-1} \boldsymbol{d}_{-1}) + 2(1 + 2\boldsymbol{d}^+_0 \boldsymbol{d}_0) =$$

$$10 + 4\boldsymbol{n_d} = 2(2\boldsymbol{n_d} + 5).$$

The third commutator reads $[\boldsymbol{P}^-\,,\, (\boldsymbol{P}^+)^n\,]$. Owing to (10.7) the relation

$$\boldsymbol{P}^-(\boldsymbol{P}^+)^n = 2(2\boldsymbol{n_d} + 5)(\boldsymbol{P}^+)^{n-1} + \boldsymbol{P}^+\boldsymbol{P}^-(\boldsymbol{P}^+)^{n-1} \tag{10.8}$$

holds. We carry the procedure on like this

$$\boldsymbol{P}^-(\boldsymbol{P}^+)^n = 2(2\boldsymbol{n_d} + 5)(\boldsymbol{P}^+)^{n-1} + \boldsymbol{P}^+\, 2(2\boldsymbol{n_d} + 5)(\boldsymbol{P}^+)^{n-2} + (\boldsymbol{P}^+)^2\boldsymbol{P}^-(\boldsymbol{P}^+)^{n-2}.$$

Going on in this way we obtain

$$\boldsymbol{P}^-(\boldsymbol{P}^+)^n = \sum_{n'=0}^{n-1} (\boldsymbol{P}^+)^{n'} 2(2\boldsymbol{n_d} + 5)(\boldsymbol{P}^+)^{n-n'-1} + (\boldsymbol{P}^+)^n \boldsymbol{P}^- =$$

$$\sum_{n'=0}^{n-1} 4(\boldsymbol{P}^+)^{n'} \boldsymbol{n_d} (\boldsymbol{P}^+)^{n-n'-1} + 10n (\boldsymbol{P}^+)^{n-1} + (\boldsymbol{P}^+)^n \boldsymbol{P}^-. \qquad (10.9)$$

With the help of (10.6) one shows the relation

$$(\boldsymbol{P}^+)^{n'} \boldsymbol{n_d} (\boldsymbol{P}^+)^{n-n'-1} = (\boldsymbol{P}^+)^{\boldsymbol{n'+1}} \boldsymbol{n_d} (\boldsymbol{P}^+)^{n-n'-2} + 2(\boldsymbol{P}^+)^{n-1} = \qquad (10.10)$$

$(\boldsymbol{P}^+)^{n-1}\, \boldsymbol{n_d} + 2(n - 1 - n')\, (\boldsymbol{P}^+)^{n-1}$ which we insert in (10.9).

$$\boldsymbol{P}^-(\boldsymbol{P}^+)^n = \sum_{n'=0}^{n-1} 4(\boldsymbol{P}^+)^{n-1} \boldsymbol{n_d} + \sum_{n'=0}^{n-1} 8(n-1)(\boldsymbol{P}^+)^{n-1} - \sum_{n'=0}^{n-1} 8n'(\boldsymbol{P}^+)^{n-1} +$$

$$10n (\boldsymbol{P}^+)^{n-1} + (\boldsymbol{P}^+)^n \boldsymbol{P}^- =$$

$$4n (\boldsymbol{P}^+)^{n-1} \boldsymbol{n_d} + 8n(n-1)(\boldsymbol{P}^+)^{n-1} - 8(1/2)(n-1)n(\boldsymbol{P}^+)^{n-1} + 10n (\boldsymbol{P}^+)^{n-1} + (\boldsymbol{P}^+)^n \boldsymbol{P}^- =$$

$$(\boldsymbol{P}^+)^{n-1}(4\, \boldsymbol{n_d} n + 4n^2 + 6n) + (\boldsymbol{P}^+)^n \boldsymbol{P}^- \qquad \text{or}$$

$$[\boldsymbol{P}^-, (\boldsymbol{P}^+)^n] = 2n(\boldsymbol{P}^+)^{n-1} \cdot (2(\boldsymbol{n_d} + n) + 3). \qquad (10.11)$$

Now we are able to look into the effect of $\boldsymbol{T}^2$ on a state of the spherical basis. Owing to (6.15) and (10.5a) we write

$$\boldsymbol{T}^2 \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle =$$

$$\boldsymbol{n_d}(\boldsymbol{n_d} + 3) \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle - \boldsymbol{P}^+\boldsymbol{P}^-\, A_{n_d\tau} (\boldsymbol{P}^+)^{(n_d - \tau)/2} \,|\, N,\, n_d = \tau,\, \tau\;\, n_\Delta\, J \,\rangle.$$

In the last ket state the number of d-bosons agrees with the seniority τ. Note that no use has been made of the explicit form of $A_{n_d\tau}$. We put

$$(n_d - \tau)/2 = n_\pi\,, \qquad (10.12)$$

rebuild our result with the help of (10.11) and take the relation $\boldsymbol{P}^- \,|\, N,\, n_d = \tau,\, \tau\, n_\Delta\, J \,\rangle = 0$ (6.31) into account as follows

$$\boldsymbol{T}^2 \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle = \boldsymbol{n_d}(\boldsymbol{n_d} + 3) \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle -$$

$$A_{n_d\tau}\, \boldsymbol{P}^+([\boldsymbol{P}^-, (\boldsymbol{P}^+)^{n_\pi}] + (\boldsymbol{P}^+)^{n_\pi} \boldsymbol{P}^-) \,|\, N,\, n_d = \tau,\, \tau\;\, n_\Delta\, J \,\rangle =$$

$$n_d(n_d + 3) \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle - A_{n_d\tau} \boldsymbol{P}^+ 2n_\pi (\boldsymbol{P}^+)^{n_\pi - 1}(2(\tau + n_\pi) + 3)\,) \,|\, N\, \tau\, \tau\, n_\Delta\, J \rangle =$$

$$(n_d(n_d + 3) - 2n_\pi(2(\tau + n_\pi) + 3) \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle = \tau\, (\tau + 3) \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle$$

where (6.15) has been inserted again. Thus, we have

$$\boldsymbol{T}^2 \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle = \tau\, (\tau + 3) \,|\, N\, n_d\, \tau\;\, n_\Delta\, J \,\rangle \qquad (10.13)$$

i. e. the states of the spherical basis are eigenfunctions of the seniority operator with the eigenvalues $\tau\, (\tau + 3)$.

From the commutation behaviour of $\boldsymbol{T}^2$, we can see also directly that eigenfunctions of $\boldsymbol{H}^{(I)}$ (10.1) are simultaneously eigenfunctions of $\boldsymbol{T}^2$. In chapters 7 and 8 it was shown that $\boldsymbol{N}$ and $\boldsymbol{J}^2$ commute with $\boldsymbol{H}$ and therefore with $\boldsymbol{H}^{(I)}$. Naturally, $\boldsymbol{n_d}$ and $\boldsymbol{H}^{(I)}$ commute with $\boldsymbol{H}^{(I)}$. According to (10.1) this operator consists essentially of the operators $\boldsymbol{N}$, $\boldsymbol{n_d}$, $\boldsymbol{J}^2$ and $\boldsymbol{T}^2$. Consequently, $\boldsymbol{T}^2$ commutes with $\boldsymbol{H}^{(I)}$. Moreover, $\boldsymbol{J}^2$ commutes with $\boldsymbol{T}^2$ which results in a common eigenfunction of the operators $\boldsymbol{H}^{(I)}$, $\boldsymbol{J}^2$ and $\boldsymbol{T}^2$.

Deriving the eigenvalue of $\boldsymbol{T}^2$ (10.13), not all properties of the many-*d*-boson state are employed. From the eigenfunction, only the number of *d*-boson pairs with $J = 0$ or the number of unpaired *d*-bosons has to be known. On the other hand E. Chacon et al. (1976, 1977) succeeded in constructing the spherical basis starting from the eigenvalue equations of $\boldsymbol{T}^2$, $\boldsymbol{J}^2$ and $\boldsymbol{J_z}$ (see also Frank (1994), p. 311).

At the end of this section, we treat the state

$$\boldsymbol{P}^+ \,|\, N, n_d - 2, \tau\ n_\Delta\, J\,\rangle,$$

of which we made use in section 6.2. By means of (10.6) and (10.7) we write

$$\boldsymbol{P}^-\boldsymbol{P}^+ = [\boldsymbol{P}^-, \boldsymbol{P}^+] + \boldsymbol{P}^+\boldsymbol{P}^- = [\boldsymbol{P}^-, \boldsymbol{P}^+] + \boldsymbol{n_d}(\boldsymbol{n_d} + 3) - \boldsymbol{T}^2 =$$

$$2(2\boldsymbol{n_d} + 5) + \boldsymbol{n_d}(\boldsymbol{n_d} + 3) - \boldsymbol{T}^2 = \boldsymbol{n_d}^2 + 7\boldsymbol{n_d} + 10 - \boldsymbol{T}^2.$$

Because of (10.13) we have

$$\boldsymbol{P}^-\boldsymbol{P}^+ \,|\, N, n_d - 2, \tau\ n_\Delta\, J\,\rangle =$$

$$((n_d - 2)^2 + 7(n_d - 2) + 10 - \tau(\tau + 3))\,|\, N, n_d - 2, \tau\ n_\Delta\, J\,\rangle =$$

$$(n_d(n_d + 3) - \tau(\tau + 3))\,|\, N, n_d - 2, \tau\ n_\Delta\, J\,\rangle.$$

The corresponding diagonal matrix element reads

$$\langle N, n_d - 2, \tau\ n_\Delta\, J \,|\, \boldsymbol{P}^-\boldsymbol{P}^+ \,|\, N, n_d - 2, \tau\ n_\Delta\, J\,\rangle = n_d(n_d + 3) - \tau(\tau + 3). \qquad (10.14)$$

The operator $\boldsymbol{P}^-$ is adjoint to $\boldsymbol{P}^+$ analogously to the operators $\boldsymbol{d_\mu}$ and $\boldsymbol{d}^+_\mu$ (see section 5.2). Consequently $\langle N, n_d - 2, \tau\ n_\Delta\, J \,|\, \boldsymbol{P}^-$ is conjugate complex with regard to the right hand side of the matrix element (10.14). Since $\boldsymbol{P}^+$ raises n_d by 2 the expression

$$(n_d(n_d + 3) - \tau(\tau + 3))^{-1/2}\, \boldsymbol{P}^+ \,|\, N, n_d - 2, \tau\ n_\Delta\, J\,\rangle = |\, N\, n_d\, \tau\ n_\Delta\, J\,\rangle \qquad (10.15)$$

is a normalised state with n_d *d*-bosons. In an analogous way we obtain

$$\langle N\, n_d\, \tau\ n_\Delta\, J \,|\, \boldsymbol{P}^+\boldsymbol{P}^- \,|\, N\, n_d\, \tau\ n_\Delta\, J\,\rangle = n_d(n_d + 3) - \tau(\tau + 3) \qquad (10.16)$$

and $\quad (n_d(n_d + 3) - \tau(\tau + 3))^{-1/2}\, \boldsymbol{P}^- \,|\, N\, n_d\, \tau\ n_\Delta\, J\,\rangle = |\, N, n_d - 2, \tau\ n_\Delta\, J\,\rangle.$

10.3 Energy eigenvalues. Comparison with experimental data

If the Hamilton operator $\boldsymbol{H}^{(\mathrm{I})}$ (10.1) acts on a state of the spherical basis (10.3) according to (10.4) and (10.13) the following eigenenergy results

$$E^{(\mathrm{I})}_{N\, n_d\, \tau\, J} = \varepsilon_n N + v_n N^2 + (\varepsilon_d{'} + v_{nd} N) n_d + v_d n_d^2 + v_t\, \tau\, (\tau + 3) + v_j\, J(J + 1). \quad (10.17)$$

Independently of the complexity of the state, the eigenvalue has a simple, closed form. Neither the projection M nor the ordering number n_Δ influence the energy but the latter restricts the range of the angular momentum J according to (10.5).

Comparing calculated with measured spectra of nuclear levels mostly one single nucleus is investigated neglecting the binding energy relative to other nuclei. The lowest level is characterised by $n_d = 0$ and therefore $\tau = J = 0$. If we put $\varepsilon_n = v_n = 0$ the energy of this state is zero. Moreover at the beginning of this chapter $v_0 = v_2 = 0$ has been chosen and so we obtain $u_0 = \varepsilon_s = 0$ (10.2) and the coefficients (10.17) read like this

$$\begin{aligned}
\varepsilon_d{'} &= \varepsilon + (3/10)c_0 - (2/7)c_2 - (18/35)c_4,\\
v_{nd} &= (1/\sqrt{5})u_2,\\
v_d &= (1/10)c_0 + (1/7)c_2 + (9/35)c_4 - (1/\sqrt{5})u_2, \qquad (10.18)\\
v_t &= -(1/10)c_0 + (1/7)c_2 - (3/70)c_4,\\
v_j &= -(1/14)c_2 + (1/14)c_4.
\end{aligned}$$

Here we have written ε instead of ε_d, as is usual. In most published comparisons with real nuclear spectra, the quantity u_2 is put equal to zero and the energies are given as follows

$$E^{(\mathrm{I})}_{n_d\, \tau\, J} = \varepsilon_d{'} n_d + v_d n_d^2 + v_t\, \tau\, (\tau + 3) + v_j\, J(J + 1). \quad (10.19)$$

Fitting theoretical on experimental values one finds that for the most part v_t and v_j are positive and small relative to $\varepsilon_d{'}$. The levels with greatest possible values for τ and J for given n_d i. e. with $\tau = n_d$ and $J = 2n_d$, constitute a conspicuous series showing the angular momenta $J =$ 0, 2, 4, 6 and so forth and slightly increasing steps in energy, which is named Y-series. Every level of this kind is the lowest one of the partial spectrum with the same angular momentum. Thus, levels of the Y-series show the following energies

$$\begin{aligned}
E^{(\mathrm{Y})}_J \equiv E^{(\mathrm{I})}_{n_d = \tau = J/2} &= (1/2)\varepsilon_d{'} J + (1/4) v_d\, J^2 + (1/2) v_t\, J(J/2 + 3) + v_j\, J(J + 1) =\\
&(1/8)[(4\varepsilon - 2c_4)J + c_4 J^2] \qquad (10.20)
\end{aligned}$$

where (10.18) and $u_2 = 0$ have been employed. Apart from the feeble quadratic term, the energy grows in (10.20) proportional to J i. e. in the same way as the eigenenergies of the harmonic oscillator. This mathematical form appears also if the atomic nucleus is interpreted as a vibrating droplet for which reason the special case dealt with in this chapter is named “vibrational limit“. On the other hand, we will show that the Hamilton operator has features of the Lie algebra $u(5)$, which is why the name “$u(5)$ limit” is also used.

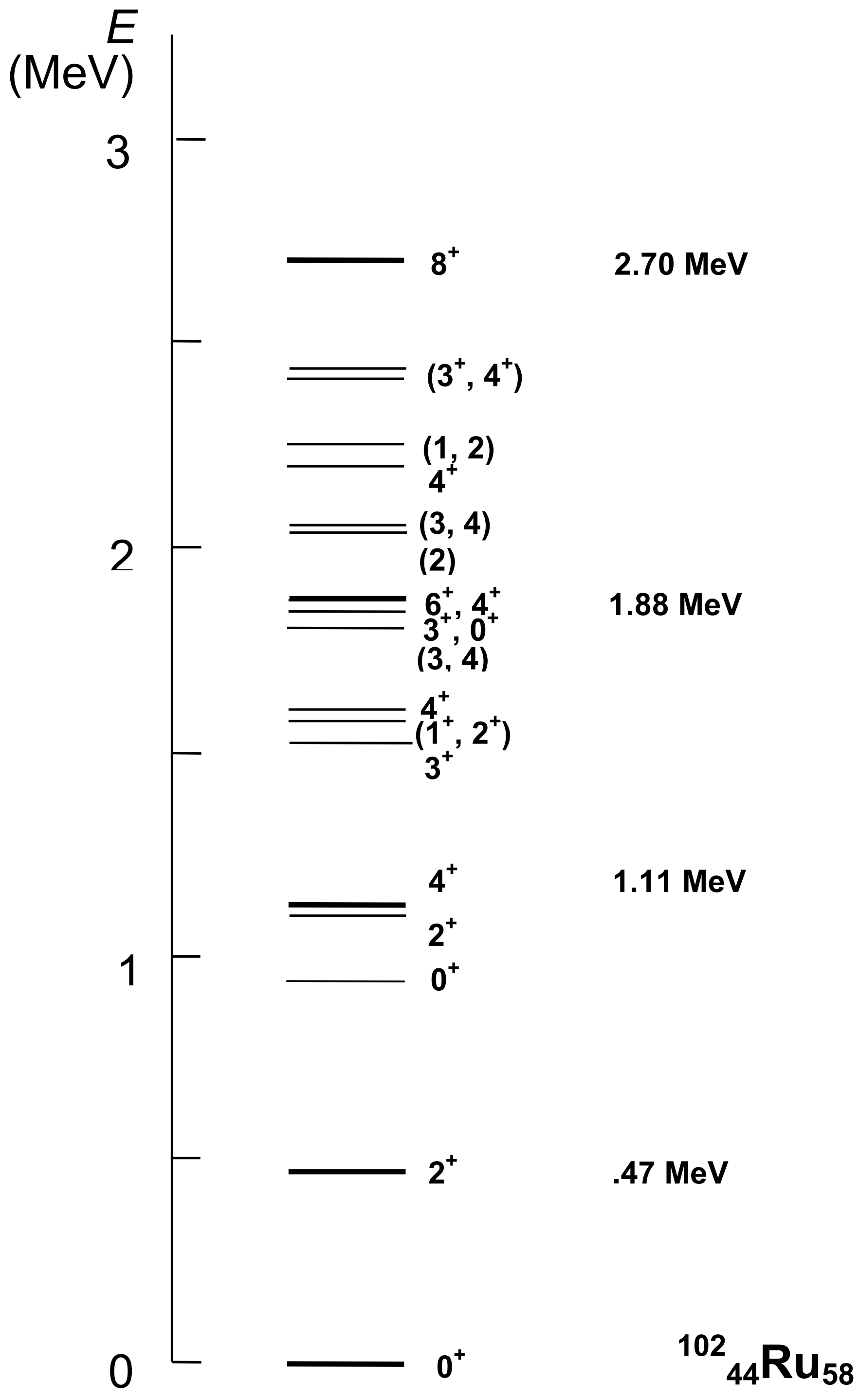

Figure 10.1. Measured spectrum of ^{102}Ru. The energies of the Y-series are given (Arima and Iachello, 1976a, p. 273).

In figure 10.1 the levels of the Y-series of the nucleus ${}^{102}{}_{44}Ru_{58}$ are marked and the energies are given. The energy difference between two levels of the Y-series following each other reads according to (10.20)

$$\Delta E^{(Y)}{}_J = E^{(Y)}{}_J - E^{(Y)}{}_{J-2} = \varepsilon - c_4 + (1/2)c_4 J. \qquad (10.21)$$

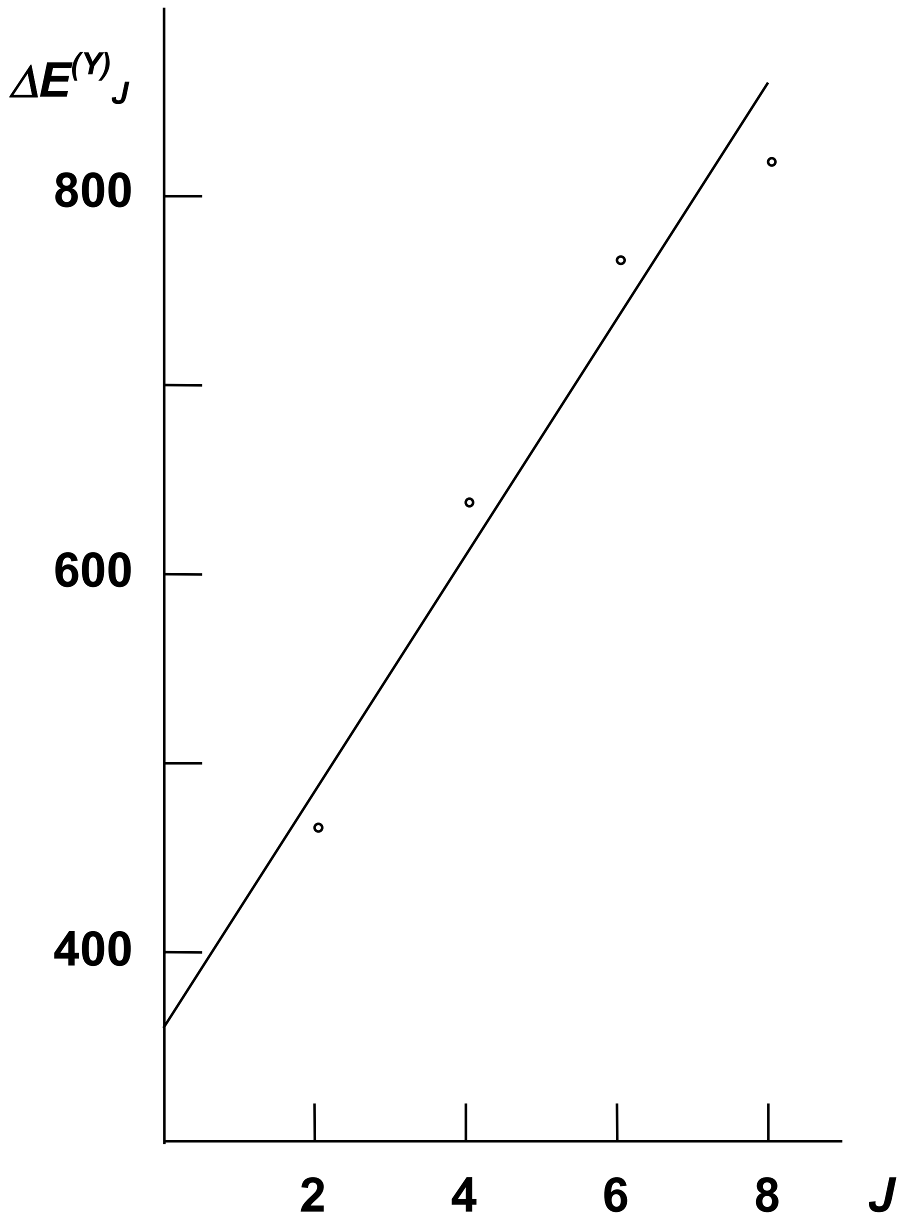

Figure 10.2. Energy differences $\Delta E^{(Y)}{}_J = E^{(Y)}{}_J - E^{(Y)}{}_{J-2}$ between levels of the Y-series following each other of the nucleus ${}^{102}{}_{44}Ru_{58}$ according to figure 10.1. The straight line fitted in by eye yields the values c_4 = 125 keV and ε = 485 keV for the coefficients in (10.21).

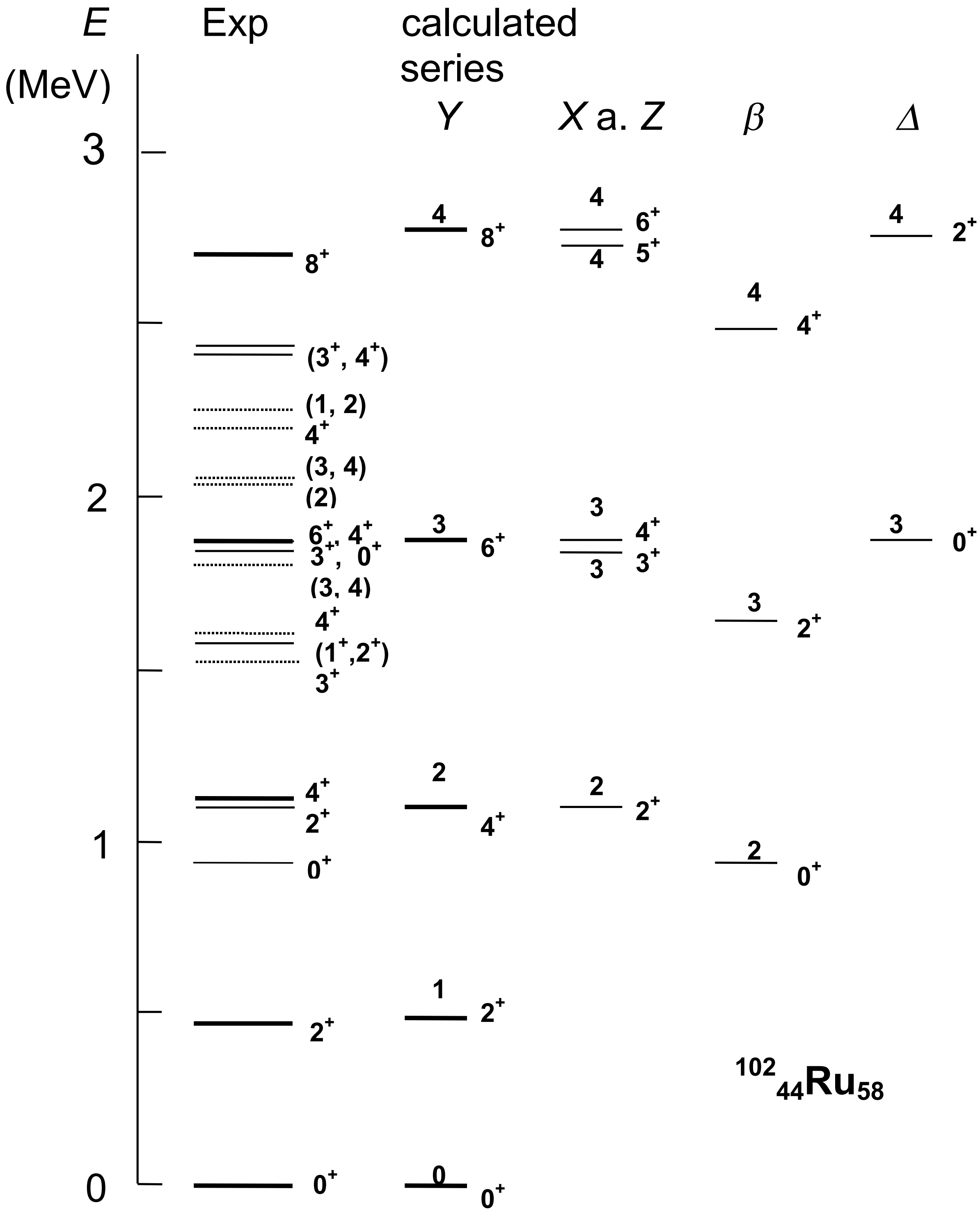

Figure 10.3. Comparison between the measured and the theoretical spectrum of ^{102}Ru. The spectrum is calculated by means of (10.19) and (10.18) with the parameters ε = 481 keV, c_0 = -18 keV, c_2 = 141 keV, c_4 = 144 keV and u_2 = 0 (Arima and Iachello, 1976a, p. 273). The numbers above or below the lines indicate the numbers of *d*-bosons.

The measured values of $\Delta E^{(Y)}_J$ of the same nucleus are plotted against J in figure 10.2. Making use of (10.21) and of a graphical procedure the coefficients c_4 and ε are found. In figure 10.3 about a dozen levels are given which have been calculated by means of (10.19) and (10.18) adjusting the coefficients c_0, c_2, c_4 and ε . The values of the last two quantities agree reasonably with those of figure 10.2. The levels are subdivided in the series Y, X, Z, β and Δ which have the following characteristics

series	τ	n_Δ	J	
Y	n_d	0	$2n_d$	
X	n_d	0	$2n_d$ - 2	
Z	n_d	0	$2n_d$ - 3	
others	n_d	0	$2n_d$ - 4	(10.22)
			$2n_d$ - 5	
β	n_d -2	0	$2\ \tau = 2n_d - 4$	
Δ	n_d	1	$2n_d$ - 6	

The rule (10.5) excludes a further series with $J = 2n_d - 1$ between the series Y and X. As in other nuclear models here it is impossible to reproduce all measured levels. Every model is based on approximations, which restrict the field of applicability.

Figures 10.4 and 10.5 show the spectra of ${}^{110}_{46}Cd_{62}$ and ${}^{188}_{78}Pt_{110}$ with the corresponding levels calculated by means of (10.19). Comparing these three measured (or calculated) spectra reveals a clear agreement of the energy structures, which is expected because (10.19) does not contain the boson number N.

Figure 10.6 depicts the series Y and X for ${}^{100}_{46}Pt_{54}$ ($N = 4$). Apparently, here the boson number has been extrapolated over the value 4 otherwise $J \leq 8$ would hold.

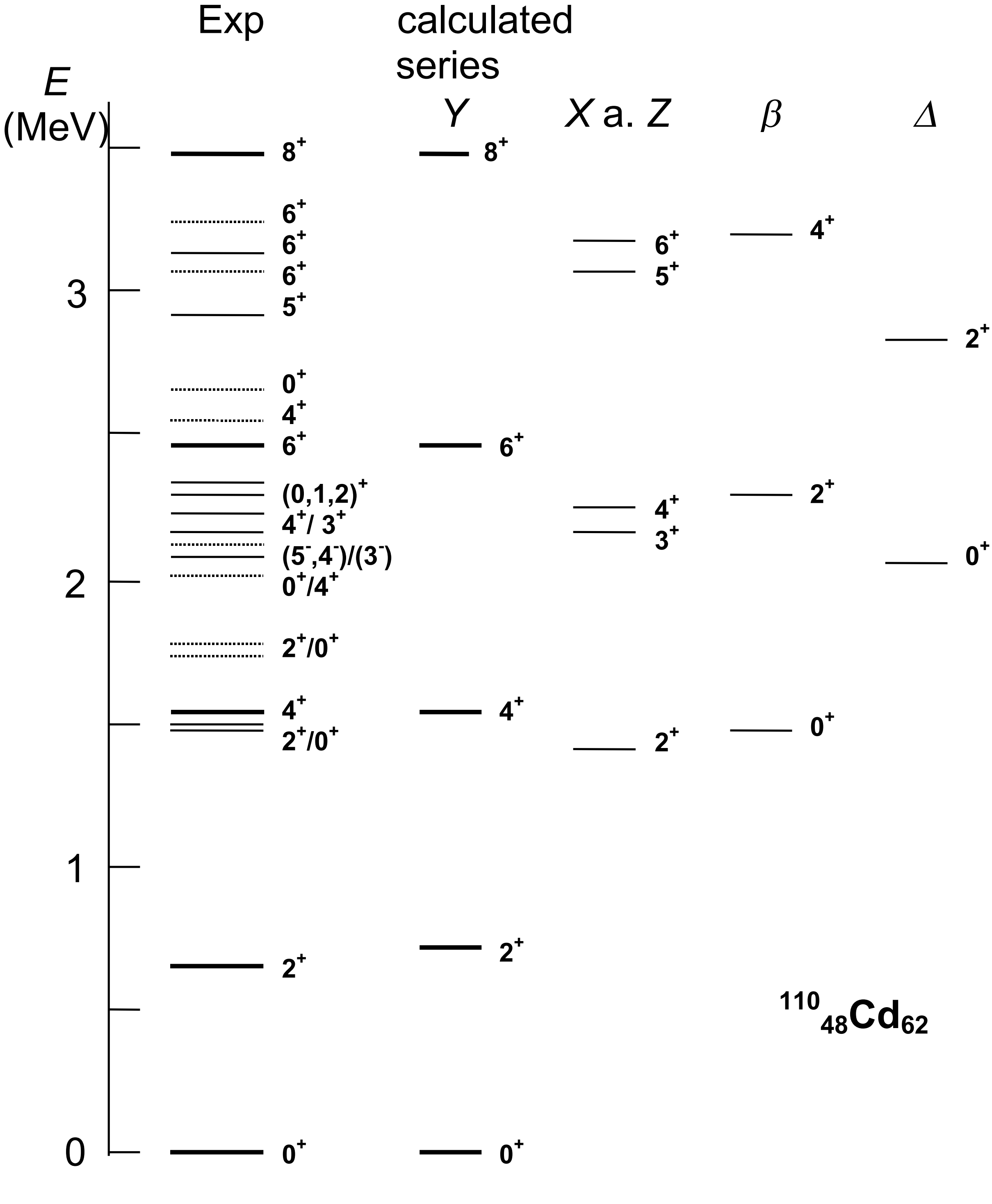

Figure 10.4. Comparison between the measured and the theoretical spectrum of ^{110}Cd. The spectrum is calculated by means of (10.19) and (10.18) with the parameters ε = 722 keV, c_0 = 29 keV, c_2 = - 42 keV, c_4 = 98 keV and $u_2 = 0$ (Arima and Iachello, 1976a, p. 274).

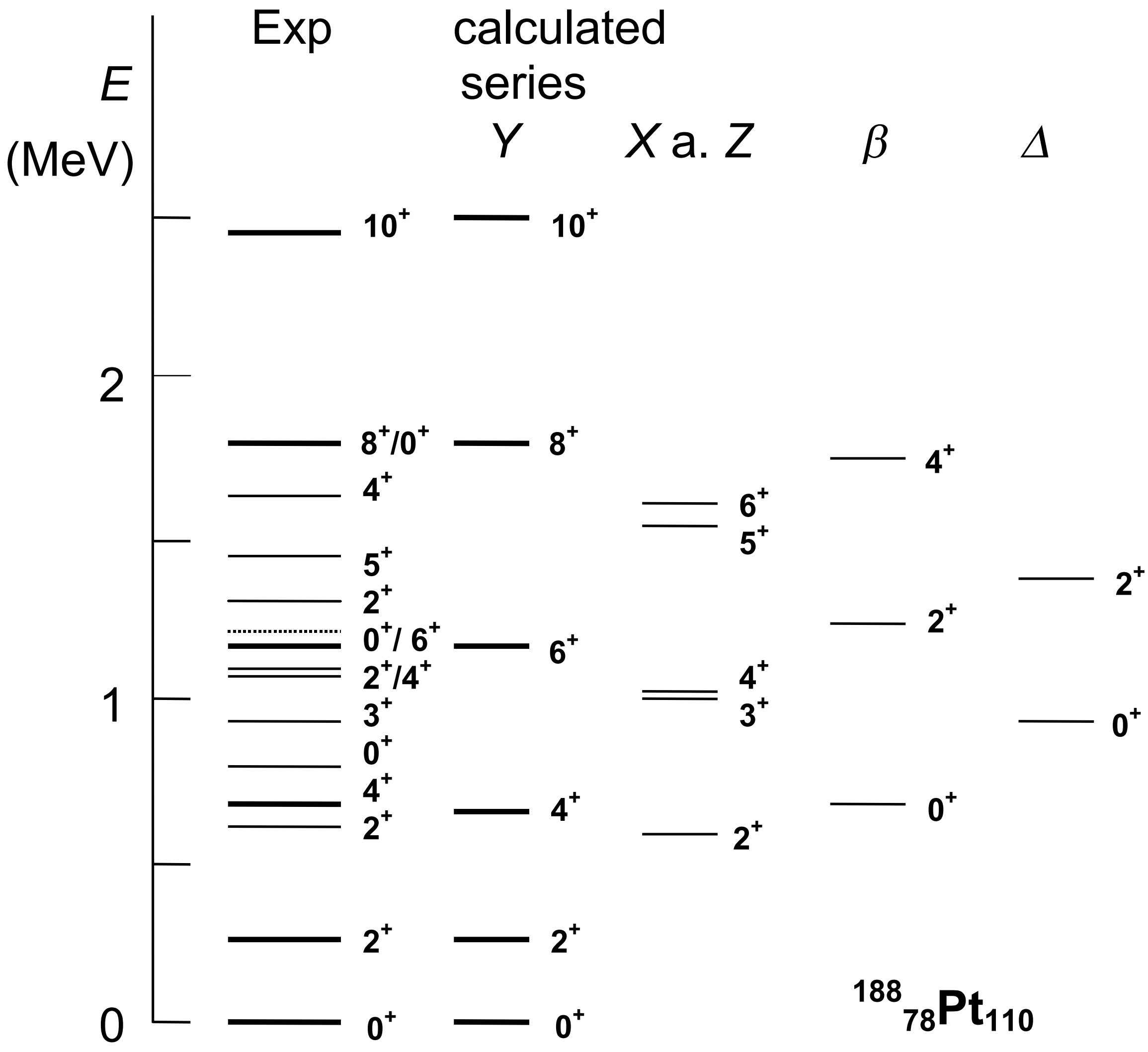

Figure 10.5. Comparison between the measured and the theoretical spectrum of ^{188}Pt. The spectrum is calculated by means of (10.19) and (10.18) with the parameters ε = 281 keV, c_0 = 148 keV, c_2 = 30 keV, c_4 = 110 keV and u_2 = 0 (Arima and Iachello, 1976a, p. 274).

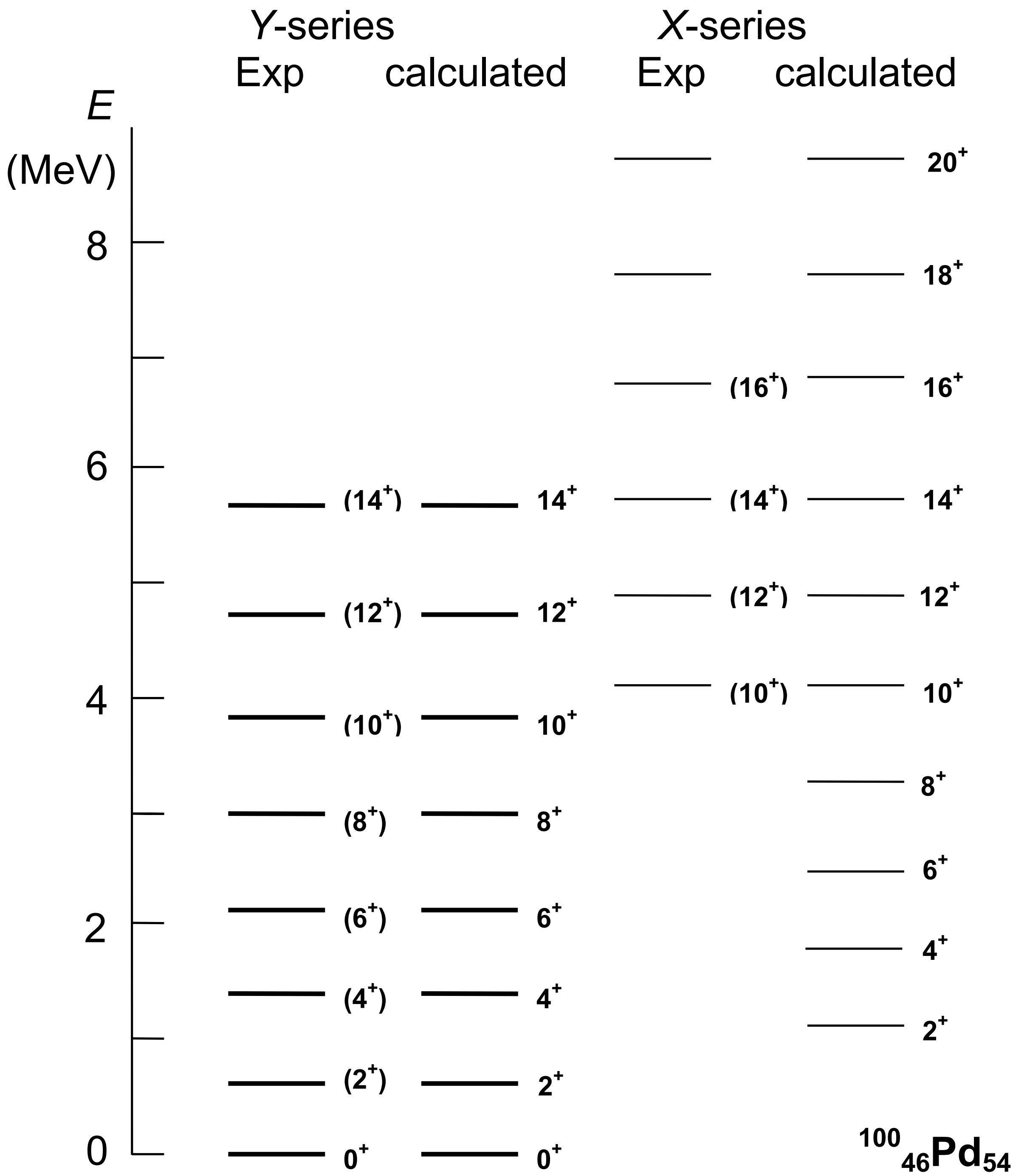

Figure 10.6. Comparison between the measured and the calculated energies of the *Y*-series and the *X*-series in ^{100}Pd. The parameters employed are $\varepsilon = 680$ keV, c_2 = -160 keV und c_4 = 45 keV (Arima and Iachello, 1976a, p. 275).

According to (7.16) the quantities c_0, c_2, c_4 and ε stand for matrix elements of interaction operators and they should be individually equal for every nucleus contrary to the reality. The discrepancy can be explained in the following way

(i) the interaction between *s*- and *d*-bosons is neglected in this limit,
(ii) the effective boson-boson interaction can depend on the total number of bosons,
(iii) no difference is made between bosons attributed to protons and those belonging to neutrons.

The limitation (i) will be dropped later for the IBM1 and the approximation (iii) is abolishes in the model IBM2.

Assigning measured to calculated levels is uncertain if no other criterion than the energy is considered. The rates of electromagnetic transitions from level to level are well-known quantities, which can help to improve the identification of states. We will deal with them in the next chapter.

Only a restricted number of nuclei behave according to the vibrational limit. Systematic investigations showed that nuclei with neutron- and/or proton numbers near the magic numbers 50, 82 and 126 belong to this limit (see figure 10.7).

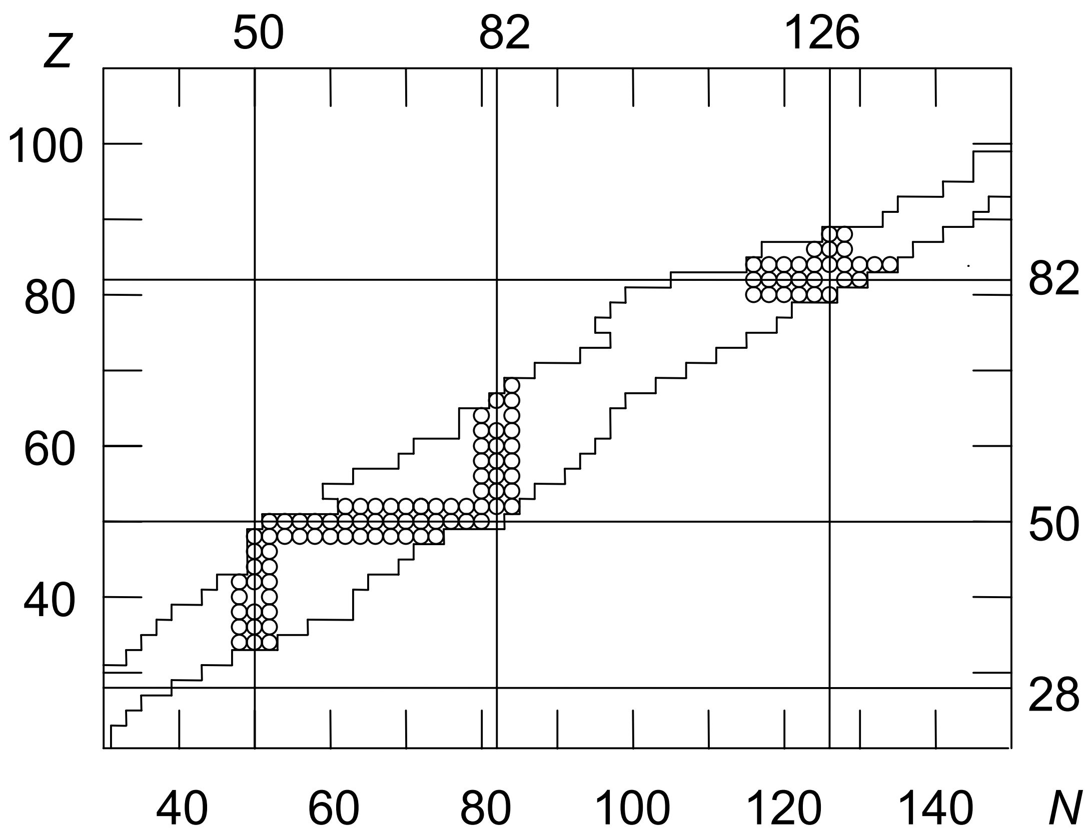

Figure 10.7. Region of the periodic table of the even-even nuclei. Z = number of protons, N = number of neutrons. The circles denote nuclei of the $u(5)$-limit (Iachello and Arima, 1987, p. 42).

11 Electromagnetic transitions in the *u*(5)-limit

Excited nuclei can reach lower energy levels by emitting electromagnetic radiation. In this chapter mainly the electric quadrupole radiation will be treated and the transition probabilities will be formulated for the IBM. First, a short introduction will be given and in the second section the definition of the interaction operator and its representation by creation- and annihilation operators for bosons follows. In the third section the connection between the matrix elements of the interaction operator and the transition probabilities will be described. The derivation of closed expressions for these matrix elements (section 11.4) and a comparison with experimental data follow. In the section 11.6 and 11.7 special transitions and levels are treated and finally we look into the quadrupole moment.

11.1 Multipole radiation

The electromagnetic field of a radiating nucleus is represented by a vector potential $\underline{A}(\underline{r})$. It is characterised by a spherical harmonic $Y_{LM}(\underline{\varrho})$ with $\underline{\varrho} = \underline{r}/r$ (Brussaard and Glaudemans, 1977, p. 179). It describes the multipole radiation of the order 2^L possessing the angular momentum L. When a nucleus emits a photon, this angular momentum is carried away from the initial nuclear state having spin J_i which results in a final state with spin J_f . Thus, the triangle condition

$$J_i + L \geq J_f \geq | J_i - L | \tag{11.1}$$

is satisfied. There are two types of multipole radiation, the electric and the magnetic. They differ not only in their spatial shape but also in the parity of the field. For example, positive parity means that two points of the field function lying diametrically with respect to the origin show the same function value. For negative parity, these values differ only in sign.

Electric multipole radiation has positive parity for even L-values and negative parity for odd ones. The magnetic multipole radiation shows just the opposite behaviour. If two fields are coupled which possess the same parity the combination shows positive parity and negative parity results if the components have different parity.

In the interacting boson model 1 and 2, single boson states (actually material waves) are coupled which have all positive parity (even angular momenta L = 0, 2 result in positive parity, which is connected with the behaviour of the spherical harmonics). Thus, all collective states of these models have positive parity. Because the emission of a photon leads from a state of positive parity to another with the same parity all types of radiation must show positive parity. Consequently, for multipole radiation the following L-values are possible

$$\begin{aligned} &\text{electric :} \quad L = 2, 4, 6, \ldots \\ &\text{magnetic :} \quad L = 1, 3, 5, \ldots \; . \end{aligned} \tag{11.2}$$

In order to formulate the probability of a radiative transition to a lower level we will looke into the energy experienced by the excited nucleus in the field of radiation i. e. into the interaction energy of the radiation field with the charges or magnetic moments of the nucleus.

11.2 The operator of the electromagnetic interaction

In quantum mechanics, the classical expression of the interaction energy is replaced by an operator. For the electric multipole radiation, this operator of the electromagnetic interaction can be written employing the long-wavelength approximation (Brussaard and Glaudemans, 1977, p. 182) like this

$$\boldsymbol{O}(\mathrm{E}, L, M) = \sum_{k=1}^{A} e(k)\, r(k)^L\, Y_{LM}(\underline{\rho}(k)). \qquad (11.3)$$

The sum covers all nucleons of the nucleus and $e(k)$ is the charge of the kth nucleon. Obviously $\boldsymbol{O}(\mathrm{E}, L, M)$ is a single nucleon operator and it corresponds formally to the single-boson operator (5.1). The operator for magnetic multipole radiation is of this type as well. Due to the spherical harmonics, both kinds of interaction operators are tensor operators (section 6.3). We transfer both properties of this operator - it is a tensor and it represents the single particle type - to the corresponding interaction operator of the interacting boson model, which we write analogously to (11.3) as follows

$$\boldsymbol{O}(\mathrm{E}, L, M) = \sum_{i=1}^{N} \boldsymbol{o}(\mathrm{E}, L, M, \underline{\xi}_i). \qquad (11.4)$$

The sum extend over all active bosons and $\boldsymbol{o}(\mathrm{E}, L, M, \underline{\xi}_i)$ is a tensor operator of the rank L. Provisionally we are attributing to the ith boson the set of variables $\underline{\xi}_i$. For the magnetic radiation an analogous relation is true. The expression (11.4) cannot be deduced completely or be described more in detail. In fact, it belongs to the basic postulates of the IBM. According to (5.25) and (5.14) the operator $\boldsymbol{O}(\mathrm{E}, L, M)$ can be written in the second quantisation as

$$\boldsymbol{O}(\mathrm{E}, L, M) = \sum_{l_p, l_q, m_p, m_q} \langle p, l_p, m_p \,|\, \boldsymbol{o}(\mathrm{E}, L, M) \,|\, q, l_q, m_q \rangle\, \boldsymbol{b}^+_{l_p\, m_p}\, \boldsymbol{b}_{l_q\, m_q}. \qquad (11.5)$$

Matrix elements of tensor operators can be split in two factors with the aid of the Wigner - Eckart theorem (A5.8) as follows

$$\langle p, l_p, m_p \,|\, \boldsymbol{o}(\mathrm{E}, L, M) \,|q, l_q, m_q \rangle =$$

$$(l_q\, m_q\, L\, M \,|\, l_p\, m_p)\, (2l_p + 1)^{-1/2} \langle p, l_p \,||\, \boldsymbol{o}(\mathrm{E}, L) \,||q, l_q \rangle. \qquad (11.6)$$

The right hand side of (11.6) contains a Clebsch-Gordan coefficient and the reduced matrix element, which is independent of the projection quantum numbers m_p, m_q and M. According to (6.19) and (A1.14) we write

$$\boldsymbol{b}^+_{l_p,\, m_p}\, \boldsymbol{b}_{l_q,\, m_q} = (-1)^{-m_q}\, \boldsymbol{b}^+_{l_p\, m_p}\, \boldsymbol{b}^{\sim}_{l_q\, m_q} =$$

$$(-1)^{-m_q} \sum_{L'\, M'} (l_p\, m_p\, l_q, -m_q \,|\, L'\, M')\, [\boldsymbol{b}^+_{l_p} \times \boldsymbol{b}^{\sim}_{l_q}]^{(L')}{}_{M'}. \qquad (11.7)$$

We insert (11.6) and (11.7) in (11.5) and obtain

$$\boldsymbol{O}(\mathrm{E}, L, M) = \sum_{L' M' l_p\, l_q} \sum_{m_p\, m_q} (-1)^{-m_q} (l_q\, m_q\, L\, M \mid l_p\, m_p)\, (l_p\, m_p\, l_q, -m_q \mid L'\, M') \cdot$$

$$(2l_p + 1)^{-1/2} \langle p, l_p \,\|\, \boldsymbol{o}(\mathrm{E},L) \,\|\, q, l_q \rangle\ [\boldsymbol{b}^{+}_{l_p} \times \boldsymbol{b}^{\sim}_{l_q}]^{(L')}{}_{M'}. \qquad (11.8)$$

Due to (A1.4) up to (A1.6) the relation

$$(-1)^{-m_q} (l_q\, m_q\, L\, M \mid l_p\, m_p) = (-1)^{l_q} ((2l_p + 1)/(2L + 1))^{1/2} (l_q\, m_q\, l_p, -m_p \mid L, -M)$$

$$= (-1)^{l_q} ((2l_p + 1)/(2L + 1))^{1/2} (l_p\, m_p\, l_q\, -m_q \mid L\, M) \qquad (11.9)$$

holds, which we insert in (11.8). Since l_q is even and because of

$\sum_{m_p\, m_q} (l_p\, m_p\ l_q, -m_q \mid L\, M)\, (l_p\, m_p\, l_q, -m_q \mid L'\, M') = \delta_{LL'}\, \delta_{MM'}$ (A1.10) we have

$$\boldsymbol{O}(\mathrm{E}, L, M) = \sum_{L' M' l_p\, l_q} \delta_{LL'}\, \delta_{MM'}\, (2L + 1)^{-1/2} \langle p, l_p \,\|\, \boldsymbol{o}(\mathrm{E},L) \,\|\, q, l_q \rangle\ [\boldsymbol{b}^{+}_{l_p} \times \boldsymbol{b}^{\sim}_{l_q}]^{(L')}{}_{M'}$$

$$= \sum_{l_p\, l_q} (2L + 1)^{-1/2} \langle p, l_p \,\|\, \boldsymbol{o}(\mathrm{E},L) \,\|\, q, l_q \rangle\ [\boldsymbol{b}^{+}_{l_p} \times \boldsymbol{b}^{\sim}_{l_q}]^{(L')}{}_{M'}. \qquad (11.10)$$

Among the radiative transitions the electric quadrupole transition is most suitable for demonstrating the nature of a collective state (Talmi, 1993, p. 746). In this case (L = 2, l_p and l_q = 0, 2) the expression (11.10) reduces to three terms

$$\boldsymbol{O}(\mathrm{E}, 2, \mu) = \alpha_2\, \boldsymbol{d}^{+}_{\mu}\, \boldsymbol{s} + \alpha_2'\, \boldsymbol{s}^{+}\, \boldsymbol{d}^{\sim}_{\mu} + \beta_2\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}{}_{\mu}$$

with
$$\alpha_2 = \langle\, d \,\|\, \boldsymbol{o}(\mathrm{E},2) \,\|\, s\, \rangle/\sqrt{5}\ ,$$

$$\alpha_2' = \langle\, s \,\|\, \boldsymbol{o}(\mathrm{E},2) \,\|\, d\, \rangle/\sqrt{5}\ ,$$

$$\beta_2 = \langle\, d \,\|\, \boldsymbol{o}(\mathrm{E},2) \,\|\, d\, \rangle/\sqrt{5}\ . \qquad (11.11)$$

Because $\boldsymbol{o}$(E,2) is a tensor operator one can show that $\alpha_2 = \alpha_2'^{*}$ holds (Talmi, 1993, p. 112). It's usual to assume that α_2 is real i. e. $\alpha_2 = \alpha_2'$ and we can write

$$\boldsymbol{O}(\mathrm{E}, 2, \mu) = \alpha_2\, (\boldsymbol{d}^{+}_{\mu}\, \boldsymbol{s} + \boldsymbol{s}^{+}\, \boldsymbol{d}^{\sim}_{\mu}) + \beta_2\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}{}_{\mu}. \qquad (11.12)$$

This is the operator of the electric quadrupole transitions of the IBM. It is not limited to a special case (limit).

We now carry out a rough estimate of the quotient β_2 / α_2 . As a model we consider a closed core with a single boson moving like a particle in the potential of the harmonic oscillator. It is represented by the function $R_l(r)\, Y_{l\,m}(\underline{\rho})$ with $\underline{\rho}$ = $\underline{r}/r$, which consists of the spherical harmonics $Y_{l\,m}(\underline{\rho})$ and of the r-dependent part $R_l(r)$ (see Lawson, 1980, p. 7). We denote the single particle operator of the quadrupole radiation according to (11.3) by $\boldsymbol{O}(\mathrm{E}, 2, M) = \boldsymbol{o}(\mathrm{E}, 2, M) = e\, r^2\, Y_{2M}(\underline{\rho})$. Thus, we write

$$\langle\, d \,\|\, \boldsymbol{o}(\mathrm{E},2) \,\|\, s\, \rangle = \mathrm{e}\, \langle\, Y_2 \,\|\, Y_2 \,\|\, Y_0\, \rangle \int R_2(r)\, R_0(r)\, r^4\, dr \text{ and}$$

$$\langle\, d \,\|\, \boldsymbol{o}(\mathrm{E},2) \,\|\, d\, \rangle = \mathrm{e}\, \langle\, Y_2 \,\|\, Y_2 \,\|\, Y_2\, \rangle \int (R_2(r))^2\, r^4\, dr. \qquad (11.13)$$

From Brussaard and Glaudemans (1977, p. 419) we take

$$\langle Y_2 \| Y_2 \| Y_0 \rangle = (25/(4\pi))^{1/2} \begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (5/(4\pi))^{1/2} \text{ and}$$

$$\langle Y_2 \| Y_2 \| Y_2 \rangle = (25/(4\pi))^{1/2} \sqrt{5} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = -5/(14\pi)^{1/2}. \tag{11.14}$$

Lawson (1980, p.7) gives the following relations

$$\int R_{2,\,n=0}(\alpha\, r)\, R_{0,\,n=0}(\alpha\, r)\, r^4\, dr = 3\sqrt{15}/(2\alpha^2) \text{ and}$$

$$\int (R_{2,\,n=0}(\alpha\, r))^2\, r^4\, dr = 7/(2\alpha^2). \tag{11.15}$$

From (11.11) up to (11.15) follows

$$\beta_2/\alpha_2 = \langle d \| \boldsymbol{o}(\mathrm{E},2) \| d \rangle / \langle d \| \boldsymbol{o}(\mathrm{E},2) \| s \rangle = -\sqrt{(14/27)} = -0.72. \tag{11.16}$$

For “physical solutions“ O. Scholten (1991, p. 99) formulates the condition $0 > \beta_2/\alpha_2 > -\sqrt{7}/2 = -1.323$, which is satisfied by (11.16). In practice the coefficients α_2 and β_2 serve as parameters, with which measured decay rates are fitted.

11.3 Transition probabilities

The theory of the electromagnetic radiation connected with quantum mechanics yields the following formula for the probability per time unit of a radiative transition from the initial state i with spin J_i to the final state f with J_f (Brussaard and Glaudemans, 1977, p. 189)

$$T(L, J_i \to J_f) = 8\pi(L+1)L^{-1}((2L+1)!!)^{-2}\,(q^{\,2L+1}/\hbar)\, B(L, J_i \to J_f). \tag{11.17}$$

L is the angular momentum of the radiation field and $(2L + 1)!!$ stands for $(2L + 1)(2L - 1)(2L - 3)\cdot \ldots \cdot 3\cdot 1$. The quantity q depends directly on the energy difference $\Delta E = E_i - E_f$ of the states involved in the transition like this

$$q = \Delta E/(\hbar c). \tag{11.18}$$

Equation (11.17) holds both for electric and magnetic multipole radiation. A multitude of nuclei being in the state i decays exponentially with the mean lifetime τ_m which is connected with $T(L, J_i \to J_f)$ as follows

$$\tau_m(L, J_i \to J_f) = 1/\,T(L, J_i \to J_f). \tag{11.19}$$

Using the uncertainty relation $\Delta E\cdot \Delta t \approx \hbar$ one can define the width of a level

$$\Gamma = \hbar/\tau_m = \hbar T \quad . \tag{11.20}$$

The quantity $B(L, J_i \to J_f)$ in (11.17) is the reduced transition probability. According to (11.17) up to (11.20) it is related in a simple way to the mean lifetime or the width of the initial level and therefore we can regard it as a measurable quantity.

From quantum mechanics we learn that $B(L,\ J_i \rightarrow J_f\ )$ is connected with the reduced matrix element of the operator $\boldsymbol{O}(L,\ M)$ of the interaction between the multipole radiation field and the nucleus as follows

$$B(L,\ J_i \rightarrow J_f) = \langle\ J_f \parallel \boldsymbol{O}(L) \parallel J_i\ \rangle^2/(2J_i + 1). \tag{11.21}$$

This relation holds for electric and magnetic transitions. The reduced matrix element $\langle\ J_f \parallel \boldsymbol{O}(L) \parallel J_i\ \rangle$ is related to the normal matrix element due to the Wigner - Eckart theorem (appendix 5) like this

$$\langle\ J_f\, M_f \mid \boldsymbol{O}(L\ M) \mid J_i\, M_i\ \rangle = (J_i\, M_i\, L\ M \mid J_f\, M_f)\ (2J_f\ + 1)^{-1/2} \langle\ J_f \parallel \boldsymbol{O}(L) \parallel J_i\ \rangle. \tag{11.22}$$

For electric radiative transitions with L = 2 we insert the expression (11.12) in the normal matrix element of (11.22) and obtain

$$\langle\ J_f\, M_f \mid \boldsymbol{O}(\mathrm{E},\ 2,\ M) \mid J_i\, M_f\ \rangle = \tag{11.23}$$

$$\langle\ n_{sf}\, n_{df}\, \tau_f\, n_{\Delta f}\, J_f\ M_f \mid \alpha_2\, (\boldsymbol{d}^+{}_{\mu}\boldsymbol{s} + \boldsymbol{s}^+\boldsymbol{d}^{\sim}{}_{\mu}) + \beta_2\, [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)}{}_{\mu} \mid n_{si}\, n_{di}\, \tau_i\, n_{\Delta i}\, J_i\ M_i\ \rangle.$$

This matrix element of the electric quadrupole transition contains three terms. The first becomes active (is different from zero) if the final state f has one d-boson more than the initial one (i), the second is attributed to the inverse situation and the third is important, if both states contain the same number of d-bosons. In (11.23) states of the spherical basis i. e. of the vibrational limit are chosen which have a definite number n_d of d-bosons. In this limit radiative quadrupole transitions occur only between states which differ in the number of d-bosons by $\Delta n_d = 0$, + 1 or – 1.

11.4 Reduced matrix elements for $|\Delta n_d|$ = 1

Now we will work on the first term of the matrix element (11.23) and formulate just the reduced matrix element of the electric quadrupole radiation making use of (11.22) and dropping n_Δ. We have

$$\langle\ n_{sf} = n_{si} - 1,\ n_{df} = n_{di} + 1,\ \tau_f\, J_f \parallel \boldsymbol{d}^+\boldsymbol{s} \parallel n_{si}\, n_{di}\, \tau_i\, J_i\ \rangle =$$

$$(2J_f + 1)^{1/2}\ (J_i\, M_i\, 2\ \mu \mid J_f\, M_f)^{-1}\ \cdot$$

$$\langle\ n_{sf} = n_{si} - 1,\ n_{df} = n_{di} + 1,\ \tau_f\, J_f\ M_f \mid \boldsymbol{d}^+{}_{\mu}\boldsymbol{s} \mid n_{si}\, n_{di}\, \tau_i\, J_i\ M_i\ \rangle. \tag{11.24}$$

Due to (5.17), $\boldsymbol{s}\ |n_{si}\ n_{di}\ \tau_i\ J_i\ \ M_i\ \rangle = \sqrt{n_{si}}\ |n_{si} - 1, n_{di}\ \tau_i\ J_i\ \ M_i\ \rangle$ holds. Because the states are factorised in a part with s- and a part with d-bosons (6.18) the “integration“ over the s-part in (11.24) yields only the factor $\sqrt{n_s}$. Therefore, we have

$$\langle\ n_{sf} = n_{si} - 1,\ n_{df} = n_{di} + 1,\ \tau_f\, J_f \parallel \boldsymbol{d}^+\boldsymbol{s} \parallel n_{si}\, n_{di}\, \tau_i\, J_i\ \rangle =$$

$$\sqrt{n_{si}}\,\langle\ n_{df} = n_{di} + 1,\ \tau_f\, J_f \parallel \boldsymbol{d}^+ \parallel n_{di}\, \tau_i\, J_i\ \rangle. \tag{11.25}$$

Temporarily we are concentrating on reduced matrix elements and leave n_s out according to the element on the right hand side of (11.25). We increase n_d by steps of 1 and begin with $n_{di} = 0$. In order to obtain $\langle\, n_{df} = 1, J_f = 2 \,||\, \boldsymbol{d}^+ \,||\, \rangle$ we form

$$\langle\, n_{df} = 1, J_f = 2, M_f \,|\, \boldsymbol{d}^+_\mu \,|\, n_{di} = 0, J_i = 0, 0 \,\rangle = \delta_{M_f \mu} \,,$$

which results from the fact that the operator $\boldsymbol{d}^+_\mu$ generates the state $|\, n_d = 1, J = 2, \mu \,\rangle$ on the right hand side. Owing to (11.22), the relations

$$1 = \langle\, n_{df} = 1, J_f = 2, \mu \,|\, \boldsymbol{d}^+_\mu \,|\, n_{di} = 0, J_i = 0, 0 \,\rangle =$$

$$(0\ 0\ 2\ \mu \,|\, 2\ \mu)\, 5^{-1/2} \langle\, n_{df} = 1,\ J_f = 2 \,||\, \boldsymbol{d}^+ \,||\, n_{di} = 0 \,\rangle, \text{ i. e.}$$

$$\langle\, n_{df} = 1,\ J_f = 2 \,||\, \boldsymbol{d}^+ \,||\, n_{di} = 0 \,\rangle = \sqrt{5} \tag{11.26}$$

hold because the Clebsch-Gordan coefficient is 1.

We now turn to $n_{di} = 1$ and $n_{df} = 2$. First we calculate the normal matrix element of $\boldsymbol{d}^+$ like this

$$\langle\, n_{df} = 2, J_f\, M_f \,|\, \boldsymbol{d}^+_\mu \,|\, n_{di} = 1, J_i = 2, M_i \,\rangle = \langle\, n_{df} = 2, J_f\, M_f \,|\, \boldsymbol{d}^+_\mu \boldsymbol{d}^+_{M_i} |\rangle. \tag{11.27}$$

Using (A1.14) and (6.5), we write

$$\langle\, n_{df} = 2, J_f\, M_f \,|\, \boldsymbol{d}^+_\mu \,|\, n_{di} = 1, J_i = 2, M_i \,\rangle =$$

$$\sum_{JM} (2\ \mu\ 2\ M_i \,|\, J\ M) \langle\, n_{df} = 2, J_f\, M_f \,|[\, \boldsymbol{d}^+ \times \boldsymbol{d}^+ \,]^{(J)}_M \,|\rangle =$$

$$\sum_{JM} (2\ \mu\ 2\ M_i \,|\, J\ M)\, \sqrt{2}\, \langle\, n_{df} = 2, J_f\, M_f \,|\, n_d = 2, J\ M \,\rangle.$$

According to the orthonormality of the n_d=2-states and of (A1.4) we obtain

$$\langle\, n_{df} = 2, J_f\, M_f \,|\, \boldsymbol{d}^+_\mu \,|\, n_{di} = 1, J_i = 2, M_i \,\rangle =$$

$$\sqrt{2}\, (2\ M_i\, 2\ \mu \,|\, J_f M_f), \quad J_f \text{ even.} \tag{11.29}$$

Moreover, according to (11.22) we have

$$\langle\, n_{df} = 2, J_f\, M_f \,|\, \boldsymbol{d}^+_\mu \,|\, n_{di} = 1, J_i = 2, M_i \,\rangle =$$

$$(2\ M_i\ 2\ \mu \,|\, J_f\, M_f\,)(2J_f + 1)^{-1/2} \langle\, n_{df} = 2,\ J_f \,||\, \boldsymbol{d}^+ \,||\, n_{di} = 1,\ J_i = 2 \,\rangle \text{, which}$$

yields

$$\langle\, n_{df} = 2,\ J_f \,||\, \boldsymbol{d}^+ \,||\, n_{di} = 1, J_i = 2 \,\rangle = \sqrt{2}\ \sqrt{(2J_f + 1)}. \tag{11.30}$$

We now take $n_{df} = 3$ and $n_{di} = 2$ i. e. we will calculate $\langle\, n_{df} = 3,\ J_f \,||\, \boldsymbol{d}^+ \,||\, n_{di} = 2, J_i \,\rangle$. We restrict ourselves to final states with the highest seniority $\tau_f = n_{df} = 3$. According to (3.9) or (10.5) this means that we are interested in values $J_f = 3$, 4 and 6. First, we work on the final state employing (6.9) and (3.7)

$$| n_{df} = 3, J' J_f M_f \rangle = A \sqrt{(3/2)} (-1)^{J_f} [\boldsymbol{d}^+ \times [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J')}]^{(J_f)}{}_{M_f} |\rangle =$$

$$[1 + 2(2J' + 1)\{{}^{2}_{2}\,{}^{2}_{J_f}\,{}^{J'}_{J'}\}]^{-1/2} \sqrt{(1/2)} (-1)^{J_f} [\boldsymbol{d}^+ \times [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J')}]^{(J_f)}{}_{M_f} |\rangle. \qquad (11.31)$$

Making use of (A3.6) and of the symmetries of the 6-*j* symbol, we write

$$| n_{df} = 3, J' J_f M_f \rangle =$$

$$[1 + 2(2J' + 1)(-1)^{J_f} \mathrm{W}(J' J_f\, 2\, J' ; 2\, 2)]^{-1/2} \sqrt{(1/2)} (-1)^{J_f} [\boldsymbol{d}^+ \times [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J')}]^{(J_f)}{}_{M_f} |\rangle.$$

We restrict ourselves to $J' = 4$, because with this choice all values for J_f mentioned above can be reached. Moreover, we employ (A1.1) and (6.5) like this

$$| n_{df} = 3, J' = 4, J_f M_f \rangle =$$

$$[1 + 2{\cdot}9\, (-1)^{J_f} \mathrm{W}(4\, J_f\, 2\, 4 ; 2\, 2)]^{-1/2} (-1)^{J_f} \cdot \sum_{\mu'} (2\, \mu', J' = 4, M_f - \mu' \,|\, J_f M_f) \cdot$$

$$\boldsymbol{d}^+_{\mu'} | n_d = 2, J' = 4, M_f - \mu' \rangle. \qquad (11.32)$$

Now we calculate the normal (not reduced) matrix element and represent the bra part with the help of (11.32) employing the hermitian adjoint operator $\boldsymbol{d}_{\mu'}$ this way

$$\langle n_{df} = 3, J' = 4, J_f M_f \,|\, \boldsymbol{d}^+_{\mu} \,|\, n_{di} = 2, J_i M_i \rangle =$$

$$[1 + 2{\cdot}9\, (-1)^{J_f} \mathrm{W}(4\, J_f\, 2\, 4 ; 2\, 2)]^{-1/2} (-1)^{J_f} \cdot \sum_{\mu'} (2\, \mu', J' = 4, M_f - \mu' \,|\, J_f M_f) \cdot$$

$$\langle n_d = 2, J' = 4, J_f, M_f - \mu' \,|\, \boldsymbol{d}_{\mu'} \boldsymbol{d}^+_{\mu} | \, n_{di} = 2, J_i, M_i \rangle \qquad (11.33)$$

Making use of $\boldsymbol{d}_{\mu'} \boldsymbol{d}^+_{\mu} = \delta_{\mu' \mu} + \boldsymbol{d}^+_{\mu} \boldsymbol{d}_{\mu'}$ and of the orthonormality of the states $| n_d = 2, J, M \rangle$ we obtain

$$\langle n_{df} = 3, J' = 4, J_f M_f \,|\, \boldsymbol{d}^+_{\mu} \,|\, n_{di} = 2, J_i M_i \rangle =$$

$$[1 + 2{\cdot}9\, (-1)^{J_f} \mathrm{W}(4\, J_f\, 2\, 4 ; 2\, 2)]^{-1/2} (-1)^{J_f} \cdot$$

$$[(2\, \mu, 4, M_f - \mu \,|\, J_f M_f)\, \delta_{J_i,4} + \sum_{\mu'} (2\, \mu', 4, M_f - \mu' \,|\, J_f M_f) \cdot$$

$$\langle n_d = 2, J' = 4, J_f, M_f - \mu' \,|\, \boldsymbol{d}^+_{\mu} \boldsymbol{d}_{\mu'} \,|\, n_{di} = 2, J_i, M_i \rangle] \qquad (11.34)$$

We look into the right hand side of the last matrix element making use of (6.5) and (A1.1) as follows

$$\boldsymbol{d}_{\mu'} \,|\, n_{di} = 2, J_i, M_i \rangle =$$

$$2^{-1/2} \sum_{\mu''} (2\, \mu''\, 2, M_i - \mu'' \,|\, J_i M_i)\, \boldsymbol{d}_{\mu'} \boldsymbol{d}^+_{\mu''} \boldsymbol{d}^+_{M_i - \mu''} |\rangle. \qquad (11.35)$$

Because of $\boldsymbol{d}_{\mu'} \boldsymbol{d}^+_{\mu''} \boldsymbol{d}^+_{M_i - \mu''} = \delta_{\mu' \mu''} \boldsymbol{d}^+_{M_i - \mu''} + \boldsymbol{d}^+_{\mu''} \delta_{\mu', M_i - \mu''} + \boldsymbol{d}^+_{\mu''} \boldsymbol{d}^+_{M_i - \mu''} \boldsymbol{d}_{\mu'}$ we have

$$\boldsymbol{d}_{\mu'} \mid n_{di} = 2, J_i, M_i\rangle = 2^{-1/2}\,(2\ \mu'\ 2, M_i - \mu' \mid J_i\ M_i)\mid d_{\boldsymbol{M}i-\mu'}\rangle +$$

$$2^{-1/2}\,(2, M_i - \mu', 2\ \mu' \mid J_i\ M_i)\mid d_{\boldsymbol{M}i-\mu'}\rangle + 0. \qquad (11.36)$$

The summand zero results from the annihilation operator acting on the vacuum state. Because J_i is even, we can write

$$\boldsymbol{d}_{\mu'} \mid n_{di} = 2, J_i, M_i\rangle = 2^{1/2}\,(2\ \mu'\ 2, M_i - \mu' \mid J_i\ M_i)\mid d_{\boldsymbol{M}i-\mu'}\rangle. \qquad (11.37)$$

Now we insert (11.37) in (11.34) and obtain

$$\langle\, n_{df} = 3, J' = 4, J_f\, M_f \mid \boldsymbol{d}^+_{\mu} \mid n_{di} = 2, J_i\, M_i\,\rangle =$$

$$[1 + 2{\cdot}9\,(-1)^{J_f}\,\mathrm{W}(4\ J_f\ 2\ 4\ ;\ 2\ 2)]^{-1/2}\,(-1)^{J_f}\cdot$$

$$[(2\ \mu, 4, M_f - \mu \mid J_f\ M_f)\ \delta_{J_i,4} + \textstyle\sum_{\mu'}(2\ \mu', 4, M_f - \mu' \mid J_f\ M_f)\cdot$$

$$\sqrt{2}\,(2\ \mu'\ 2, M_i - \mu' \mid J_i\ M_i)\,\langle\, n_d = 2, J' = 4, J_f, M_f - \mu' \mid \boldsymbol{d}^+_{\mu} \mid d_{\boldsymbol{M}i-\mu'}\,\rangle \qquad (11.38)$$

For the last matrix element, we take the expression (11.29). Moreover, we make use of the Wigner-Eckart theorem (A5.8) this way

$$\langle\, n_{df} = 3, J' = 4, J_f\, M_f \mid \boldsymbol{d}^+_{\mu} \mid n_{di} = 2, J_i\, M_i\,\rangle =$$

$$[1 + 2{\cdot}9\,(-1)^{J_f}\,\mathrm{W}(4\ J_f\ 2\ 4\ ;\ 2\ 2)]^{-1/2}\,(-1)^{J_f}\cdot$$

$$[(2\ \mu, 4, M_f - \mu \mid J_f\ M_f)\ \delta_{J_i,4} + \textstyle\sum_{\mu'}(2\ \mu', 4, M_f - \mu' \mid J_f\ M_f)\cdot$$

$$\sqrt{2}\,(2\ \mu'\ 2, M_i - \mu' \mid J_i\ M_i)\,\sqrt{2}\,(2, M_i - \mu', 2\ \mu \mid J' = 4, M_f - \mu')] =$$

$$(J_i\ M_i\ 2\ \mu \mid J_f\ M_f)\,(2J_f + 1)^{-1/2}\,\langle\, n_{df} = 3, J_f\ \|\ \boldsymbol{d}^+\ \|\ n_{di} = 2, J_i\,\rangle. \qquad (11.39)$$

We multiply both sides of the last equation with (J_i M_i 2 μ | J_f M_f) and sum over μ and M_i. Making use of (A1.10), (A1.4) and (A3.4) we obtain.

$$\langle\, n_{df} = 3, J_f \,\|\, \boldsymbol{d}^+ \,\|\, n_{di} = 2, J_i\,\rangle = [1 + (-1)^{J_f}{\cdot}2{\cdot}9{\cdot}W(4\ J_f\ 2\ 4\ ;\ 2\ 2)]^{-1/2}\cdot$$

$$(-1)^{J_f}(2J_f + 1)^{1/2}\,[(-1)^{J_f}\,\delta_{J_i,4} + 6\sqrt{(2J_i + 1)}\ W(J_i\ 2\ J_f\ 4;\ 2\ 2)],\quad (J_i\ \text{even}). \qquad (11.40)$$

Thus, we have represented the reduced matrix element of $\boldsymbol{d}^+$ on the basis of the vibrational limit with maximal seniority up to n_{di} = 2 (n_{df} = 3). Arima and Iachello (1976a, p. 269) have carried on the procedure up to n_{di} = 3 and have elaborated a closed algebraic expression depending from n_{di} . For n_{di} = 0, 1 or 2 it agrees with our relations (11.26), (11.30) and (11.40).

The reduced matrix element of the second operator of the electric quadrupole transition (11.23) can be written analogously to (11.25) as

$$\langle\, n_{sf} = n_{si} + 1, n_{df} = n_{di} - 1, J_f \,\|\, \boldsymbol{s}^+\boldsymbol{d}^{\sim} \,\|\, n_{si}\, n_{di}\, J_i\,\rangle =$$

$$\sqrt{(n_{si} + 1)}\,\langle\, n_{df} = n_{di} - 1,\ J_f \,\|\, \boldsymbol{d}^{\sim} \,\|\, n_{di}\, J_i\,\rangle. \qquad (11.41)$$

With the help of (A5.8) and (A1.4) up to (A1.6) one obtains

$\langle\, n_d - 1, J' \,\|\, \boldsymbol{d}^{\sim} \,\|\, n_d\, J \,\rangle =$

$$\sqrt{(2J' + 1)}\, (J\, M\, 2\, \mu \mid J'\, M')^{-1} \langle\, n_d - 1,\ J'\, M' \mid \boldsymbol{d}^{\sim}_{\boldsymbol{\mu}} \mid n_d\, J\, M \,\rangle =$$

$$\sqrt{(2J' + 1)}\, (J\, M\, 2\, \mu \mid J'\, M')^{-1} (-1)^{\mu} \langle\, n_d\, ,\, J\, M \mid \boldsymbol{d}^{+}_{-\boldsymbol{\mu}} \mid n_d - 1,\ J'\, M' \,\rangle =$$

$$\sqrt{(2J + 1)}\, (J'\, M'\, 2\, -\mu \mid J\, M\,)^{-1} (-1)^{J - J'} \langle\, n_d\, ,\, J\, M \mid \boldsymbol{d}^{+}_{-\boldsymbol{\mu}} \mid n_d - 1,\ J'\, M' \,\rangle =$$

$$(-1)^{J - J'} \langle\, n_d\, ,\ J \,\|\, \boldsymbol{d}^{+} \,\|\, n_d - 1,\ J' \,\rangle. \qquad (11.42)$$

In the equations (11.41) and (11.42), n_d is not limited. They are true as well if the seniority τ does not agree with n_d .

If the radiative transition of the type $n_{di} \to n_{df} = n_{di} + 1$, which appears in (11.24) up to (11.40), is energetically forbidden the inverse process $n_{di} \to n_{df} = n_{di} - 1$ may be possible which is described in (11.42) down to (11.40).

The reduced probability for transitions between states of the Y-series in the vibrational limit (section 10.3) can be written now for an arbitrary boson number n_d. Emitting E2-radiation the state $|\, n_s\, n_d,\ J_i = 2n_d = M_i \,\rangle$ turns into $|\, n_s + 1,\, n_d - 1,\, J_f = 2(n_d - 1) = M_f \,\rangle$. The reduced matrix element reads according to (11.41), (11.42) and (A5.8) as follows

$$\langle\, n_{sf} = n_{si} + 1,\, n_{df} = n_{di} - 1,\ \ J_f \,\|\, \boldsymbol{s}^{+}\boldsymbol{d}^{\sim} \,\|\, n_{si}\, n_{di}\, J_i \,\rangle =$$

$$\sqrt{(n_{si} + 1)}\, \sqrt{(2J_f + 1)}\, (J_i\, J_i\, 2,\, -2 \mid J_f\, J_f\,)^{-1} \cdot$$

$$\langle\, n_{di} - 1,\ \ J_f,\ M_f = J_f\ \mid \boldsymbol{d}_{\boldsymbol{2}} \mid n_{di}\ J_i\, M_i = J_i \,\rangle\, (-1)^{-2}.$$

Because of $J_f = J_i - 2$ we have

$$(J_i\, J_i\, 2,\, -2 \mid J_f\, J_f\,) = \sqrt{(2J_f + 1)}\, (2\ 2\ J_f\ J_f \mid J_i\, J_i) / \sqrt{(2J_i + 1)} =$$

$$\sqrt{(2J_f + 1)} / \sqrt{(2J_i + 1)}.$$

The initial state is “stretched“ and we have set it in the z-direction and so it consists only on d_2-single states. Therefore the operator $\boldsymbol{d}_{\boldsymbol{2}}$**,** which appears above, transforms the initial state directly in the final one and adds the factor $\sqrt{n_d}$, from which follows

$$\langle\, n_{si} + 1,\, n_{di} - 1,\ \ J_f \,\|\, \boldsymbol{s}^{+}\boldsymbol{d}^{\sim} \,\|\, n_{si}\, n_{di}\, J_i \,\rangle = \sqrt{(n_{si} + 1)}\, \sqrt{(2J_i + 1)}\, \sqrt{n_{di}}\, . \qquad (11.43)$$

Owing to (11.21) and (11.23) we can write

$$B(\mathrm{E2},\, J_i \to J_f\,) = (\alpha_2)^2 \langle\, n_{si} + 1,\, n_{di} - 1,\ \ J_f \,\|\, \boldsymbol{s}^{+}\boldsymbol{d}^{\sim} \,\|\, n_{si}\, n_{di}\, J_i \,\rangle / (2J_i + 1) =$$

$$(\alpha_2)^2\, (n_{si} + 1)\, n_{di\,.} \qquad (11.44)$$

Especially for the lowest transition in the Y-series the following relation holds

$$B(\mathrm{E2},\, 2^{+}{}_1 \to 0^{+}{}_1\,) = (\alpha_2)^2 (N - 1 + 1) \cdot 1 = (\alpha_2)^2 N. \qquad (11.45)$$

11.5 Comparison with experimental data of electric quadrupole transitions

In figures 10.3 up to 10.6 the values for n_d are 0, 1, 2, .. . On the right hand side of each Y-level there are levels with about the same energy which agree in number n_d with the corresponding Y-level (see also 10.22). Consequently for energy reasons transitions from n_d to $n_d - 1$ prevail. Measurements reveal that transitions with $\Delta n_d = 0$ are much weaker and an ascent from n_d to $n_d + 1$ is practically excluded. For transitions in the Y-series the expressions, (11.43) and (11.44) can be applied. According to (11.41) and (11.42) generally holds

$$\langle\, n_{sf} = n_{si} + 1,\, n_{df} = n_{di} - 1,\;\; J_f \,||\, \boldsymbol{s}^{+} \boldsymbol{d}^{\sim} \,||\, n_{si}\, n_{di}\, J_i \,\rangle =$$

$$\sqrt{}(n_{si} + 1)\ (-1)^{J_i - J_f} \langle\, n_{di}\;\; J_i \,||\, \boldsymbol{d}^{+} \,||\, n_{di} -1,\; J_f \,\rangle$$

where the last reduced matrix element is calculated by means of (11.26), (11.30) or (11.40) according to the number n_{di} . Doing so the indices i and f have to be interchanged. From (11.21), one obtains the reduced transition probability.

In table 11.1 transitions of the type $n_d \rightarrow n_d - 1$ are listed. The reduced matrix elements ME and the transition probabilities B are given. They are compared with measured values of $^{110}{}_{48}Cd_{62}$. The model nucleus contains 7 active bosons. In view of the sensitivity of the quantity B (quadratic expression) these results corroborate the theory and the interpretations of the involved levels. Measured reduced probabilities B for transitions with $|\, n_d \,| > 1$ are two orders of magnitude smaller than the values of table 11.1. In our model these transitions are inadmissible. For the transition E2, $2_1 \rightarrow 0_1$ Milner (1969) gives the measured value $B(\mathrm{E2},\, 2_1 \rightarrow 0_1)$ = 934 (+/- 38) $e^2\ fm^4$ which yields using (11.45)

$$\alpha_2 = 11{,}6 \text{ e fm}^2. \tag{11.46}$$

Table 11.1. Comparison between calculated and measured B(E2) values for the nucleus ^{110}Cd .

Initial - state spin*	n_d	series	final state spin*	n_d	series	ME^2	$B/(\alpha_2)^2$	in-dex	quotients of reduced transition probabilities	calcul.	exper.**
2_1^+	1	Y	0_1^+	0	Y	5	7	a			
4_1^+	2	Y	2_1^+	1	Y	18	12	b	B(b)/B(a)	1.71	1.53 (.19)
2_2^+	2	X	2_1^+	1	Y	10	12	c	B(c)/B(a)	1.71	1.08 (.29)
3_1^+	3	Z	2_2^+	2	X	15	5·15/7	d			
3_1^+	3	Z	4_1^+	2	Y	6	5·6/7	e	B(e)/B(d)	0.4	0.47 (.2)
4_2^+	3	X	2_2^+	2	X	11	5·11/7	f			
4_2^+	3	X	4_1^+	2	Y	10	5·10/7	g	B(g)/B(f)	0.91	0.23 (.3)

The series and the levels correspond to figure 10.4. The quantity ME is the reduced matrix element of the d-boson states according to (11.42).
* The indices beside the spin value serves as spectroscopic identification.
** The estimated errors (Arima and Iachello, 1976a, p. 287) are put in parentheses. The data a up to c stem from Milner (1969) and d up to g are taken from Krane and Steffen (1970).

Table 11.2 shows further quotients of reduced transition probabilities compared with measurements.

Table 11.2. Reduced transition probabilities for quadrupole radiation of Xe-isotopes (Arima and Iachello, 1976a, p. 276).

Quotients of the B's	calculated	measured values			
		$^{132}_{54}Xe_{78}$	$^{130}_{54}Xe_{76}$	$^{126}_{54}Xe_{72}$	$^{124}_{54}Xe_{74}$
$B(3_1^+\rightarrow 4_1^+)$ / $B(3_1^+\rightarrow 2_2^+)$	0.4	0.51	0.24-0.25	0.46-0.72	0.16
$B(4_2^+\rightarrow 4_1^+)$ / $B(4_2^+\rightarrow 2_2^+)$	0.91		0.95-1.05	0.94	0.90

The measured values of transition probabilities which are forbidden in the IBM1 are two orders of magnitude smaller.

11.6 Transitions with $|\Delta n_d| = 0$

In our treatment of the quadrupole radiation the transitions with unchanged numbers of d-bosons (Δn_d = 0, i. e. $n_{di} = n_{df}$) were left over. In order to make them up, first we look into the normal matrix element of the part of the transition operator in question in (11.23) and write with the aid of (11.22)

$$(J_i\, M_i\, 2\, m \mid J_f\, M_f)\, (2J_f + 1)^{-1/2} \langle\, n_d\, \alpha_f\, J_f \,||\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \,||\, n_d\, \alpha_i\, J_i \,\rangle =$$

$$\langle\, n_d\, \alpha_f\, J_f\, M_f \mid [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}{}_m \mid n_d\, \alpha_i\, J_i\, M_i \,\rangle =$$

$$\sum_{\mu\,\mu'} (2\, \mu\, 2\, \mu' \mid 2\, m) \langle\, n_d\, \alpha_f\, J_f\, M_f \mid \boldsymbol{d}^{+}{}_{\mu} \boldsymbol{d}^{\sim}{}_{\mu'} \mid n_d\, \alpha_i\, J_i\, M_i \,\rangle. \qquad (11.47)$$

The specification α comprises n_Δ and the seniority τ. We now use the concept of the completeness of state functions of a quantum mechanical system. For spatial functions $\varphi_b(\underline{r})$ of this kind the following equation holds

$$\sum_\beta \varphi_\beta(\underline{r})\, \varphi_\beta{}^*(\underline{r}') = \delta(\, \underline{r} - \underline{r}'\,).$$

Consequently the functions Φ_i and Φ_f which belong to the same system obey the relation

$$\int \Phi_f^*(\underline{r})\, \Phi_i(\underline{r})\, d\underline{r} \equiv \iint \Phi_f^*(\underline{r})\, \delta(\, \underline{r} - \underline{r}'\,) \Phi_i(\underline{r}')\, d\underline{r}\, d\underline{r}' =$$

$$\sum_\beta \int \Phi_f^*(\underline{r})\, \varphi_\beta(\underline{r})\, d\underline{r} \cdot \int \varphi_\beta{}^*(\underline{r}')\, \Phi_i(\underline{r}')\, d\underline{r}'.$$

Thus, we can write down the last matrix element of (11.47) this way

$$\langle\, n_d\, \alpha_f\, J_f\, M_f \mid \boldsymbol{d}^{+}{}_{\mu} \boldsymbol{d}^{\sim}{}_{\mu'} \mid n_d\, \alpha_i\, J_i\, M_i \,\rangle =$$

$$\sum_{\alpha\, J\, M} \langle\, n_d\, \alpha_f\, J_f\, M_f \mid \boldsymbol{d}^{+}{}_{\mu} \mid n_d - 1,\, \alpha\, J\, M \,\rangle \langle\, n_d - 1,\, \alpha\, J\, M \mid \boldsymbol{d}^{\sim}{}_{\mu'} \mid n_d\, \alpha_i\, J_i\, M_i \,\rangle =$$

$$\sum_{\alpha\, J\, M} (J\, M\, 2\, \mu \mid J_f\, M_f)\, (2J_f + 1)^{-1/2}\, (J_i\, M_i\, 2\, \mu' \mid J\, M)\, (2J + 1)^{-1/2} \cdot$$

$$\langle\, n_d\, \alpha_f\, J_f \,||\, \boldsymbol{d}^{+} \,||\, n_d - 1,\, \alpha\, J \,\rangle \langle\, n_d - 1,\, \alpha\, J \,||\, \boldsymbol{d}^{\sim} \,||\, n_d\, \alpha_i\, J_i \,\rangle. \qquad (11.48)$$

In the last line the Wigner-Eckart theorem (A5.8) or (11.22) has been applied twice. We now insert (11.48) in (11.47), multiply by $(J_i\, M_i\, 2\, m \mid J_f\, M_f)$, sum over m (and M_i) and obtain by means of (A1.10), (A3.4) and (A3.6) the relation

$$\langle\, n_d\, \alpha_f\, J_f \,||\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \,||\, n_d\, \alpha_i\, J_i \,\rangle =$$

$$\sum_{m\,\mu\,\mu'} (J_i\, M_i\, 2\, m \mid J_f\, M_f)\, (2\, \mu\, 2\, \mu' \mid 2\, m) \cdot$$

$$\sum_{\alpha\, J\, M} (J\, M\, 2\, \mu \mid J_f\, M_f)\, (J_i\, M_i\, 2\, \mu' \mid J\, M)\, (2J + 1)^{-1/2} \cdot \qquad (11.49)$$

$$\langle\, n_d\, \alpha_f\, J_f \,||\, \boldsymbol{d}^{+} \,||\, n_d - 1,\, \alpha\, J \,\rangle \langle\, n_d - 1,\, \alpha\, J \,||\, \boldsymbol{d}^{\sim} \,||\, n_d\, \alpha_i\, J_i \,\rangle =$$

$$(-1)^{J_i - J_f} \sqrt{5} \sum_{\alpha\, J} \{{}^{2}{}_{J_i}\, {}^{2}{}_{J}\, {}^{2}{}_{J_f}\} \langle\, n_d\, \alpha_f\, J_f \,||\, \boldsymbol{d}^{+} \,||\, n_d - 1,\, \alpha\, J \,\rangle \langle\, n_d - 1,\, \alpha\, J \,||\, \boldsymbol{d}^{\sim} \,||\, n_d\, \alpha_i\, J_i \,\rangle.$$

As an example, we calculate the matrix element of the E2-transition from the level 4_1 to 2_2. According to figures 10.3 up to 10.5 these states are the lowest one with $n_d = 2$ and maximal seniority. Since there is only one level with $n_d = 1$ (2_1-level) the expression (11.49) contains here merely one summand as follows

$$\langle\, n_d = 2,\, J_f = 2 \,||\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} \,||\, n_d\, J_i = 4 \,\rangle =$$

$$\sqrt{5}\, \{{}^{2}{}_{4}\, {}^{2}{}_{2}\, {}^{2}{}_{2}\} \langle\, n_d = 2,\, J_f = 2 \,||\, \boldsymbol{d}^{+} \,||\, n_d = 1,\, J = 2 \,\rangle \cdot$$

$$\langle\, n_d = 1,\, J = 2 \,||\, \boldsymbol{d}^{\sim} \,||\, n_d = 2,\, J_i = 4 \,\rangle = \sqrt{5}\, (2/35)\, \sqrt{10}\, \sqrt{18} = 12/7. \quad (11.50)$$

The values of the last matrix elements are taken from table 11.1 and the 6-j symbol comes from (A3.14).

The sum in (11.49) extends over all states with n_d - 1 including possibly higher levels than (n_d, α_i, J_i) or lower ones than (n_d, α_f, J_f). Therefore, the sum can comprise levels not having maximal seniority, which means that the prerequisite of (11.40) is not met.

11.7 Configurations with $n_d > \tau$

With that, we turn to quadrupole transitions whose states have not maximal seniority and belong therefore to the series β or other according to (10.22). Such transitions have relatively low probabilities in nuclei of the vibrational limit but anyway we will need the affiliated matrix element for the treatment of the general Hamilton operator of the IBM1 (chapter 12).

I. Talmi (1993, p. 766 - 775) has shown in an easily readable fashion that matrix elements of the kind

$$\langle\, n_d',\ \tau' \leq n_d',\ n_\Delta',\ J' \,||\, \boldsymbol{d}^+ ||\, n_d = n_d' - 1,\ \tau \leq n_d,\ n_\Delta,\ J \,\rangle \qquad (11.51)$$

are connected by simple factors with matrix elements having maximal seniority in both included states (the effect of $\boldsymbol{d}^+$ on the state on the right hand side increases τ by 1 or lowers it by 1 because an J=0-pair can arise). The corresponding matrix element of the operator $\boldsymbol{d}^{\sim}$ is formulated by means of (11.42).

In an analogous manner, I.Talmi has treated the matrix element

$$\langle\, n_d,\ \tau' \leq n_d,\ n_\Delta',\ J' \,||\, [\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)} ||\, n_d,\ \tau \leq n_d,\ n_\Delta,\ J \,\rangle$$

where $\tau = \tau' - 2$, τ' or $\tau' + 2$ holds.

11.8 Quadrupole moments

Electric multiple moments are measures of the charge distribution of nuclear states and especially of their deviation from spherical symmetry. The quadrupole moment Q_z conveys how the charge is arranged along the z-axis. Classically Q_z is defined like this

$$Q_z = \sum_{k=1}^{A} e(k)\,(3z(k)^2 - r(k)^2). \qquad (11.52)$$

A is the number of nucleons and $e(k)$ is the charge of the kth nucleon. The spherical harmonics of the second order read

$$r^2\, Y_{2\,0}(\underline{r}) = \sqrt{(5/16\pi)}\,(3z^2 - r^2).$$

We insert them in (11.52) and obtain

$$Q_z = \sqrt{(16\pi/5)} \sum_{k=1}^{A} e(k)\, r(k)^2\, Y_{2\,0}(\underline{r}(k)). \qquad (11.53)$$

In quantum mechanics this function act as an operator, which we name $\boldsymbol{Q}^{(2)}$ ($\equiv Q_z$). From (11.3) we see that $\boldsymbol{Q}^{(2)}$ agrees largely with the operator $\boldsymbol{O}$(E, 2, 0) i. e.

$$\boldsymbol{Q}^{(2)} = \sqrt{(16\pi/5)}\ \boldsymbol{O}(\mathrm{E}, 2, 0). \qquad (11.54)$$

According to (11.4) this operator results in a single-boson operator. Analogously to (11.12) it is written by means of creation- and annihilation operators like this

$$\boldsymbol{Q}^{(2)} = \sqrt{(16\pi/5)}\ (\ \alpha_2 (\boldsymbol{d}^+_0\, \boldsymbol{s} + \boldsymbol{s}^+ \boldsymbol{d}^\sim_0) + \beta_2 [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_0\ ). \qquad (11.55)$$

In the vibration limit the operator $\boldsymbol{d}^+_0\, \boldsymbol{s} + \boldsymbol{s}^+\, \boldsymbol{d}^\sim_0$ does not contribute to the expectation value of $\boldsymbol{Q}^{(2)}$ because this operator combines only boson states with different numbers n_d (Δn_d = + 1 or - 1). The quadrupole moment $Q^{(2)}$ of the nuclear state $|\ n_d\ \alpha\ J\ M\ \rangle$ is defined as the expectation value of $\boldsymbol{Q}^{(2)}$ where the nuclear spin is put in the z-axis i. e. $M = J$. Therefore this moment reads

$$Q^{(2)} = \sqrt{(16\pi/5)}\ \beta_2 \langle\ n_d\ \alpha\ J,\ M = J\ |\ [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_0\ |\ n_d\ \alpha\ J,\ M = J\ \rangle. \qquad (11.56)$$

The specification α is the same as in (11.47). In order to obtain consistent expressions we generalise the concept of the reduced probability of the electric quadrupole radiation (11.23, 11.21). Starting from (11.21), (11.12) and (11.22), we define

$$(B(\mathrm{E2}, (n_d\ \alpha\ J)\rightarrow(n_d\ \alpha\ J)))^{1/2} = (2J + 1)^{-1/2}\ \beta_2 \langle\ n_d\ \alpha\ J\ ||[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}||\ n_d\ \alpha\ J\ \rangle =$$

$$(J\ J\ 2\ 0\ |\ J\ J)^{-1}\ \beta_2 \langle\ n_d\ \alpha\ J,\ M = J\ |\ [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_0\ |\ n_d\ \alpha\ J,\ M = J\ \rangle. \qquad (11.57)$$

Because of $(J\ J\ 2\ 0\ |\ J\ J) = \sqrt{J}\ (2J - 1)^{1/2}(J + 1)^{-1/2}\ (2J + 3)^{-1/2}$ and from (11.56) and (11.57) we have the following relation

$$Q^{(2)} = \qquad (11.58)$$

$$\sqrt{(16\pi/5)}\ \sqrt{J}\ (2J - 1)^{1/2}(J + 1)^{-1/2}\ (2J + 3)^{-1/2}\ (B(\mathrm{E2}, (n_d\ \alpha\ J)\rightarrow(n_d\ \alpha\ J)))^{1/2}.$$

This expression shows that calculating $Q^{(2)}$, one can use the formalism developed for B.

Now we calculate quantitatively the quadrupole moment of the level 2_1 in the vibrational limit of the IBM1. According to (11.58), (11.57) and (11.21) we write

$$Q^{(2)}(2_1) = \sqrt{(16\pi/5)}\ \sqrt{(2/7)}\ \sqrt{(1/5)} \langle\ n_d = 1,\ J = 2\ ||[\ \boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}||\ n_d = 1,\ J = 2\ \rangle.$$

With the help of (11.49), (11.26), (11.42) and (A3.14) we continue like this

$$Q^{(2)}(2_1) = \sqrt{(16\pi/5)}\ \sqrt{(2/7)}\ \sqrt{(1/5)}\ \beta_2\ \sqrt{5}\ \{^2{}_2\ {}^2{}_0\ {}^2{}_2\} \cdot$$

$$\langle\ n_d = 1,\ J = 2\ ||\ \boldsymbol{d}^+\ ||\ \rangle \langle\ ||\ \boldsymbol{d}^\sim ||\ n_d = 1,\ J = 2\ \rangle =$$

$$\sqrt{(16\pi/5)}\ \sqrt{(2/7)}\ \beta_2 = 1.695\ \beta_2. \qquad (11.59)$$

One obtains the same result from (11.56) using the relation

$$[\, \boldsymbol{d}^{+} \times \tilde{\boldsymbol{d}}\,]^{(2)}_{0} = \sum_{\mu} (2\ \mu\ 2\ {-\mu} \mid 2\ 0)\ \boldsymbol{d}^{+}_{\mu} \tilde{\boldsymbol{d}}_{-\mu} = \sum_{\mu} (\mu^{2} - 2)/\sqrt{14}\ \boldsymbol{d}^{+}_{\mu} \boldsymbol{d}_{\mu}\,. \qquad (11.60)$$

12 The treatment of the complete Hamiltonian of the IBM1

In chapter 10 the Hamiltonian has been reduced to the form of the vibrational limit. In this chapter we go back to the complete Hamiltonian of the IBM1 (7.17 or 9.39). In the next section the eigenstates of this Hamilton operator will be developed on the basis of the eigenfunctions of the vibrational limit i. e. on the spherical basis. The well-known diagonalisation procedure is outlined. To this end, the matrix elements of the Hamiltonian have to be formed on the spherical basis, which is brought up in the second section. In the following section the electric quadrupole radiation is treated in this model. Finally comparisons with measured nuclear states and a hint for a roughly simplified model will be given.

12.1 Eigenstates

According to (9.39) and (10.1) the Hamilton operator of the IBM1 can be written as

$$\boldsymbol{H} = \boldsymbol{H}^{(\mathbf{I})} + v_r \boldsymbol{R}^{\mathbf{2}} + v_q \boldsymbol{Q}^2 \text{ with}$$

$$\boldsymbol{H}^{(\mathbf{I})} = \varepsilon_n N + v_n N^2 + (\varepsilon_d{}' + v_{nd} N)\boldsymbol{n_d} + v_d \boldsymbol{n_d}^2 + v_t \boldsymbol{T}^2 + v_j \boldsymbol{J}^2. \quad (12.1)$$

In section 10.2 we have shown that the states of the spherical basis, $|\, N\, n_d\, \tau\, n_\Delta\, J\, M\,\rangle$, are eigenfunctions of the operator $\boldsymbol{H}^{(\mathbf{I})}$ with the eigenvalues $E^{(\mathrm{I})}{}_{N\, nd\, \tau\, J}$ (10.17). The orthogonality and the completeness of this basis, which have been used in section 11.6, make it possible to represent functions satisfying the same boundary conditions by expanding them in series of basis functions. Therefore, the eigenfunctions of $\boldsymbol{H}$, which are numbered by the index l (small letter of L), can be written like this

$$|\, l\, N\, J\, M\,\rangle\rangle = \textstyle\sum_{i=1}{}^{Z} c^{(l)}{}_{N\,J}(i)\, |\, i\, N\, J\, M\,\rangle. \quad (12.2)$$

The index i represents the triple of the ordering numbers $n_{d,}$ τ and n_Δ of the basis states. The sum comprises Z states of this kind. The coefficients $c^{(l)}{}_{N\,J}(i)$ normalise the state $|\, l\, N\, J\, M\,\rangle\rangle$ to 1.

The eigenenergy of the l th state (12.2) is characterized by $E^{(l)}{}_{N\,J}$ and satisfies the relation

$$\boldsymbol{H}\, |\, l\, N\, J\, M\,\rangle\rangle = E^{(l)}{}_{N\,J}\, |\, l\, N\, J\, M\,\rangle\rangle\, . \quad (12.3)$$

Here we insert (12.2) and have

$$\textstyle\sum_{i=1}{}^{Z} c^{(l)}{}_{N\,J}(i)\, \boldsymbol{H}\, |\, i\, N\, J\, M\,\rangle = E^{(l)}{}_{N\,J} \textstyle\sum_{i=1}{}^{Z} c^{(l)}{}_{N\,J}(i)\, |\, i\, N\, J\, M\,\rangle. \quad (12.4)$$

By multiplying with $\langle\, k\, N\, J\, M\,|$ and by integrating we obtain

$$\textstyle\sum_{i=1}{}^{Z} c^{(l)}{}_{N\,J}(i)\, \langle\, k\, N\, J\, M\,|\, \boldsymbol{H}\,|\, i\, N\, J\, M\,\rangle = E^{(l)}{}_{N\,J}\; c^{(l)}{}_{N\,J}(k). \quad (12.5)$$

We name the matrix element above $H_{NJM}(k\,i)$. It will be treated in section 12.2. The system of Z linear, homogeneous equations

$$\sum_{i=1}^{Z} H_{NJM}(k\,i)\, c^{(l)}_{NJ}(i) = E^{(l)}_{NJ}\, c^{(l)}_{NJ}(k), \quad k = 1, \dots, Z \tag{12.6}$$

is soluble only if the determinant of the matrix of this system vanishes i. e.

$$|\mathsf{H}_{NJM} - \mathsf{1} \cdot E^{(l)}_{NJ}| = 0. \tag{12.7}$$

The matrix H_{NJM} contains the elements $H_{NJM}(k\,i)$ and the matrix $\mathsf{1}$ is the unit matrix. The secular equation (12.7) yields Z roots $E^{(l)}_{NJ}$ (l = 1, ... , Z), which are the energy eigenvalues of the complete Hamilton operator for N bosons with the total angular momentum J. By inserting $E^{(l)}_{NJ}$ in (12.6) one obtains the affiliated eigenvector ($c^{(l)}_{NJ}(1), \dots, c^{(l)}_{NJ}(Z)$), which produces the state $|\, l\,N\,J\,M\,\rangle\rangle$ according to (12.2). It can be shown that the matrix H_{NJM} of the Hamilton operator is diagonalised by the matrix of the eigenvectors, for which reason this method is named diagonalisation procedure.

12.2 Matrix elements of the Hamiltonian

Now we shall calculate the matrix elements of (12.5), which are constituted by states of the spherical basis. Remember that the indices i and k denote triples of quantum numbers. Consequently, i (or k) can be written as a function $i(n_d, \tau, n_\Delta)$. According to (12.5), (12.1) and (10.17) we write

$$\langle\, k\,N\,J\,M \,|\, \boldsymbol{H} \,|\, i\,N\,J\,M\,\rangle = \langle\, k\,N\,J\,M \,|\, \boldsymbol{H}^{(\mathbf{l})} + v_r \boldsymbol{R}^2 + v_q \boldsymbol{Q}^2 \,|\, i\,N\,J\,M\,\rangle =$$

$$[\varepsilon_n N + v_n N^2 + (\varepsilon_d{}' + v_{nd} N) n_d + v_d n_d^2 + v_t\, \tau\,(\tau + 3) + v_j J(J + 1)]\, \delta_{ik} +$$

$$v_r \langle\, k\,N\,J\,M \,|\, \boldsymbol{R}^2 \,|\, i\,N\,J\,M\,\rangle + v_q \langle\, k\,N\,J\,M \,|\, \boldsymbol{Q}^2 \,|\, i\,N\,J\,M\,\rangle. \tag{12.8}$$

We turn to the last term in (12.8) but one, i. e. to the matrix element of the operator $\boldsymbol{R}^2$ (9.3), which reads

$$\boldsymbol{R}^2 = \boldsymbol{N}(\boldsymbol{N} + 4) - (\sqrt{5}\cdot[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} - \boldsymbol{s}^+\boldsymbol{s}^+)(\sqrt{5}[\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)} - \boldsymbol{ss}).$$

Making use of (9.2) we can write

$$\boldsymbol{R}^2 = \tag{12.9}$$

$$N(N + 4) + \boldsymbol{T}^2 - \boldsymbol{n_d}(\boldsymbol{n_d} + 3) - \sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}\boldsymbol{ss} - \sqrt{5}\, \boldsymbol{s}^+\boldsymbol{s}^+[\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)} - \boldsymbol{s}^+\boldsymbol{s}^+\boldsymbol{ss}.$$

With the aid of the commutation rules for the s-bosons, of the number operator $\boldsymbol{n_s} = \boldsymbol{s}^+\boldsymbol{s}$ and (10.13) we obtain

$$\langle\, k(n_d{}'\,\tau'\,n_\Delta{}')\, N\, J\, M \,|\, \boldsymbol{R^2} \,|\, i(n_d\,\tau\,n_\Delta)\, N\, J\, M \,\rangle =$$

$$[N(n + 4) + \tau(\tau + 3) - n_d(n_d + 3) - \boldsymbol{n_s}^2 + \boldsymbol{n_s}\,]\,\delta_{i\,k} -$$

$$\sqrt{5}[\,\langle\, k\,N\,J\,M \,|\, [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} |\, i\,N\,J\,M \,\rangle\, \sqrt{n_s}\sqrt{(n_s - 1)}\;\delta_{n_d{}',n_d + 2} +$$

$$\langle\, k\,N\,J\,M \,|\, [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0} \,|\, i\,N\,J\,M \,\rangle\, \sqrt{(n_s + 1)}\sqrt{(n_s + 2)}\;\delta_{n_d{}',n_d - 2}]. \qquad (12.10)$$

We put the relation $\boldsymbol{n_s} = \boldsymbol{N} - \boldsymbol{n_d}$ and $\sqrt{5}\,[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} = \sum_\mu (-1)^\mu\, \boldsymbol{d}^+_\mu\, \boldsymbol{d}^+_{-\mu}$ in (12.10), which yields

$$\langle\, k(n_d{}'\,\tau'\,n_\Delta{}')\, N\, J\, M \,|\, \boldsymbol{R^2} \,|\, i(n_d\,\tau\,n_\Delta)\, N\, J\, M \,\rangle =$$

$$[(N - n_d)\,(2n_d + 5) + n_d + \tau(\tau + 3)]\;\delta_{i\,k} - \qquad (12.11)$$

$$\sum_\mu (-1)^\mu [\,\langle\, k(n_d{}'\,\tau'\,n_\Delta{}')\, N\,J\,M \,|\, \boldsymbol{d}^+_\mu \boldsymbol{d}^+_{-\mu} \,|\, i(n_d\,\tau\,n_\Delta)\, N\,J\,M \,\rangle\, \sqrt{n_s}\sqrt{(n_s - 1)}\;\delta_{n_d{}',n_d + 2} +$$

$$\langle\, k(n_d{}'\,\tau'\,n_\Delta{}')\, N\,J\,M \,|\, \boldsymbol{d}^\sim_\mu \boldsymbol{d}^\sim_{-\mu} |\, i(n_d\,\tau\,n_\Delta)\, N\,J\,M \,\rangle\, \sqrt{(n_s + 1)}\sqrt{(n_s + 2)}\;\delta_{n_d{}',n_d - 2}].$$

Matrix elements like $\langle\, k\; N\; J\; M \,|\, \boldsymbol{d}^+_\mu\, \boldsymbol{d}^+_{-\mu} \,|\, i\; N\; J\; M \,\rangle$ in principle have been treated in (11.48). Correspondingly the relation

$$\langle\, k\,N\,J\,M \,|\, \boldsymbol{d}^+_\mu \boldsymbol{d}^+_{-\mu} \,|\, i\,N\,J\,M \,\rangle =$$

$$\sum_{i''\,J''\,M''} \langle\, k\,N\,J\,M \,|\, \boldsymbol{d}^+_\mu |\, i''\,N\,J''\,M'' \,\rangle \langle\, i''\,N\,J''\,M'' \,|\, \boldsymbol{d}^+_{-\mu} \,|\, i\,N\,J\,M \,\rangle \qquad (12.12)$$

holds. For the operator $\boldsymbol{d}^\sim_\mu\, \boldsymbol{d}^\sim_{-\mu}$ in (12.11) an analogous expression can be written down. Thus, we have formulated completely the matrix elements of the operator $\boldsymbol{R}^2$ in (12.8).

We now turn to the last matrix element in (12.8), which contains the operator $\boldsymbol{Q}^2$. According to (9.4) it reads

$$\boldsymbol{Q}^2 = \sum_\mu (-1)^\mu\, \boldsymbol{Q}_\mu\, \boldsymbol{Q}_{-\mu}$$

with $\boldsymbol{Q}_\mu = \boldsymbol{d}^+_\mu \boldsymbol{s} + \boldsymbol{s}^+ \boldsymbol{d}^\sim_\mu - (\sqrt{7}/2)\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu$. We form

$$\boldsymbol{Q}_\mu \boldsymbol{Q}_{-\mu} = \boldsymbol{d}^+_\mu \boldsymbol{s}\, \boldsymbol{d}^+_{-\mu} \boldsymbol{s} + \boldsymbol{d}^+_\mu \boldsymbol{s}\boldsymbol{s}^+ \boldsymbol{d}^\sim_{-\mu} + \boldsymbol{s}^+\, \boldsymbol{d}^\sim_\mu\, \boldsymbol{d}^+_{-\mu} \boldsymbol{s} + \boldsymbol{s}^+\, \boldsymbol{d}^\sim_\mu\, \boldsymbol{s}^+\, \boldsymbol{d}^\sim_{-\mu} -$$

$$(\sqrt{7}/2)\, (\boldsymbol{d}^+_\mu \boldsymbol{s}\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_{-\mu} + \boldsymbol{s}^+\, \boldsymbol{d}^\sim_\mu [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_{-\mu} +$$

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu\, \boldsymbol{d}^+_{-\mu} \boldsymbol{s} + [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu\, \boldsymbol{s}^+ \boldsymbol{d}^\sim_{-\mu}\,) +$$

$$(7/4)\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_{-\mu}\,. \qquad (12.13)$$

Making use of

$$\sum_\mu (-1)^\mu (\boldsymbol{d}^+_\mu \boldsymbol{s}\boldsymbol{s}^+ \boldsymbol{d}^\sim_{-\mu} + \boldsymbol{s}^+\, \boldsymbol{d}^\sim_\mu\, \boldsymbol{d}^+_{-\mu} \boldsymbol{s}\,) = \boldsymbol{n_d}\,(\boldsymbol{n_s} + 1) + (5 + \boldsymbol{n_d}\,)\boldsymbol{n_s} = \qquad (12.14)$$

$$\boldsymbol{n_d} + 2\, \boldsymbol{n_d}\, \boldsymbol{n_s} + 5\, \boldsymbol{n_s}$$

we write

$$\langle\, k\, N\, J\, M \mid \boldsymbol{Q}^2 \mid i\, N\, J\, M \,\rangle =$$

$$\sum_{\mu} (-1)^{\mu} \langle\, k(n_d'\, \tau'\, n_\Delta')\, N\, J\, M \mid \boldsymbol{Q}_{\mu}\, \boldsymbol{Q}_{-\mu} \mid i(n_d\, \tau\, n_\Delta)\, N\, J\, M \,\rangle =$$

$$(5N + 2Nn_d - 4n_d - 2n_d^2\,)\delta_{i\,k} \; +$$

$$\sum_{\mu} (-1)^{\mu} \langle\, k(n_d'\, \tau'\, n_\Delta')\, N\, J\, M \mid [\boldsymbol{d}^{+}_{\mu}\, \boldsymbol{d}^{+}_{-\mu}\, \sqrt{n_s}\sqrt{(n_s - 1)}\; \delta_{n_d',n_d+2} + \qquad (12.15)$$

$$\boldsymbol{d}^{\sim}_{\mu}\, \boldsymbol{d}^{\sim}_{-\mu}\, \sqrt{(n_s + 1)}\, \sqrt{(n_s + 2)}\; \delta_{n_d',n_d-2} -$$

$$(\sqrt{7}/2)\sqrt{n_s}\, (\boldsymbol{d}^{+}_{\mu}\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{-\mu} + [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{\mu}\, \boldsymbol{d}^{+}_{-\mu}\,)\; \delta_{n_d',n_d+1} -$$

$$(\sqrt{7}/2)\sqrt{(n_s + 1)}\; (\boldsymbol{d}^{\sim}_{\mu} [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{-\mu} + [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{\mu}\, \boldsymbol{d}^{\sim}_{-\mu})\delta_{n_d',n_d-1} +$$

$$(7/4)\; [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{\mu}\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{-\mu}\; \delta_{n_d',n_d}\;] \mid i(n_d\, \tau\, n_\Delta)\, N\, J\, M \,\rangle\; .$$

The first two terms in the matrix element on the right hand side of (12.15) have to be treated by means of (12.12). The next two terms contain matrix elements of the type

$$\langle\, k\, N\, J\, M \mid \boldsymbol{d}^{+}_{\mu}\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{-\mu} \mid i\, N\, J\, M \,\rangle = \qquad (12.16)$$

$$\sum_{i''\,J''\,M''} \langle\, k\, N\, J\, M \mid \boldsymbol{d}^{+}_{\mu} \mid i''\, N\, J''\, M'' \,\rangle \langle\, i''\, N\, J''\, M'' \mid [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{-\mu} \mid i\, N\, J\, M \,\rangle.$$

Similarly, the last term on the right hand side of (12.15) is written as

$$\langle\, k\, N\, J\, M \,|[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{\mu}\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{-\mu}] \mid i\, NJM \rangle \;\; = \qquad (12.17)$$

$$\sum_{i''\,J''\,M''} \langle\, k\, N\, J\, M \,|[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{\mu} \mid i''\, N\, J''\, M'' \,\rangle \langle\, i''\, N\, J''\, M'' \mid [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{-\mu} \mid i\, N\, J\, M \,\rangle.$$

Making use of $[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}_{\mu} = \sum_{m} (2\; m\; 2,\; \mu - m \mid 2\; \mu)\; \boldsymbol{d}^{+}_{m}\, \boldsymbol{d}^{\sim}_{\mu-m}$ the remaining terms in (12.16) and (12.17) can be calculated by means of (11.48). The matrix elements of the operators $\boldsymbol{d}^{+}_{\mu}$ and $\boldsymbol{d}^{\sim}_{\mu}$ are reviewed in section 11.4. Matrix elements of complex *d*-boson configurations are calculated with the help of the coefficients of fractional parentage (Talmi, 1993, S. 763, 766 and Bayman, 1966), which have been mentioned shortly in section 6.2. Evidently, the numerical calculation of matrix elements and the diagonalisation can become laborious. The computer program package PHINT (appendix A7) coded by O. Scholten (1991) performs this work and calculates eigenenergies, eigenstates and reduced transition probabilities.

The method applied in this chapter is named configuration mixing because pure states of the spherical basis are linearly combined. The resulting states don't show any group theoretical symmetries in contrast to the states $|\, N\, n_d\, \tau\, n_\Delta\, J\, M \,\rangle$. For this reason, there is talk of symmetry breaking.

12.3 Electric quadrupole radiation

In order to calculate the reduced transition probability (11.21) we employ initial and final states of the type (12.2) and form the reduced matrix element of the transition operator (11.12) like this

$$\langle\langle\, I_f\, N\, J_f \;||\; \boldsymbol{O}(\mathrm{E}, 2)\;||\; I_i\;\; N\, J_i\,\rangle\rangle =$$

$$\langle\langle\, I_f\, N\, J_f \;||\;\; \alpha_2\,(\boldsymbol{d}^+\,\boldsymbol{s} + \boldsymbol{s}^+\,\boldsymbol{d}^{\sim}) + \beta_2\,[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)} ||\; I_i\;\; N\, J_i\,\rangle\rangle = \qquad (12.18)$$

$$\textstyle\sum_{k\,n}\, c^{(I_f)}{}_{N\,J_f}(k)\; c^{(I_i)}{}_{N\,J_i}(n) \cdot \langle\; k\; N\, J_f \,||\, \alpha_2\,(\boldsymbol{d}^+\boldsymbol{s} + \boldsymbol{s}^+\boldsymbol{d}^{\sim}\,) + \beta_2\,[\boldsymbol{d}^+ \times \boldsymbol{d}^{\sim}]^{(2)} ||\; n\; N\, J_i\;\rangle.$$

The last matrix elements have been treated in the normal form in section 11.4. They are transformed to the reduced form with the help of (11.6). By means of (11.17) and (11.21) the expression (12.18) is connected with the transition probability per time unit T or with the mean lifetime τ_m according to (11.19).

12.4 Comparison with experimental data

Figures 12.1 and 12.2 show comparisons between measured and calculated level energies of the nuclei ^{110}Cd and ^{102}Ru, for which the complete Hamilton operator of the IBM1 has been employed. The parameters of the simplified form (7.31) of the Hamiltonian ε, c_0, c_2, c_4, v_2 and v_o are given. The calculated level schemes of both nuclei don't differ much from the schemes of figures 10.3 and 10.4 because it is about a numerical adaptation to nuclei which belong clearly to the vibrational limit. For ^{102}Ru the calculated and measured reduced transition probabilities agree quite well (Kaup, 1983, p. 16).

Han, Chuu and Hsieh (1990) have calculated about 260 level energies of Sm-, Gd- and Dy-isotopes and about 140 transitions probabilities B(E,2), which they compared with measured values. By adjusting the boson number N as a free parameter, an essential improvement could be achieved compared with earlier publications.

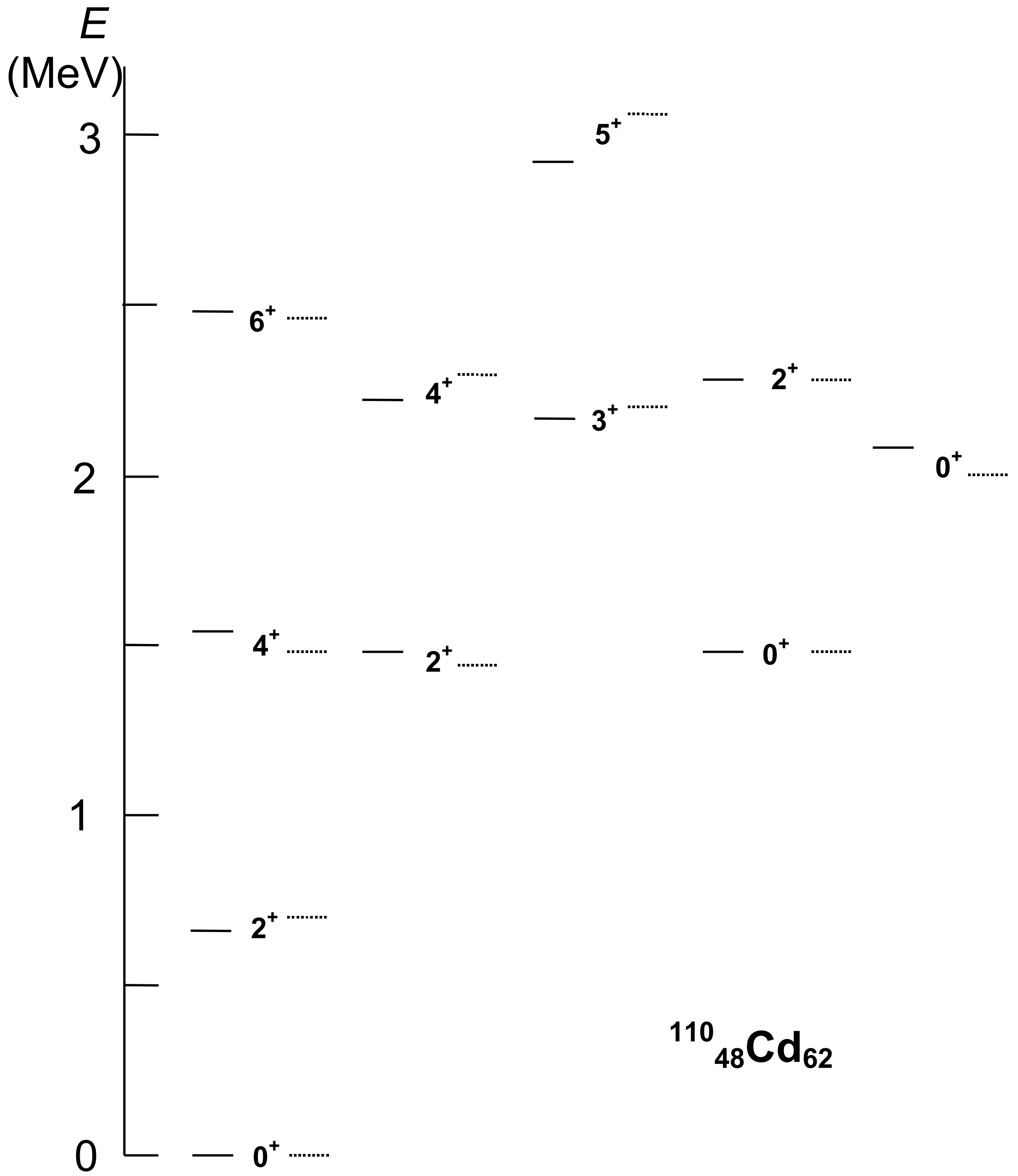

Figure 12.1. Comparison between the measured and the theoretical spectrum of ^{110}Cd. It is calculated with the parameters ε = 740 keV, c_0 = 30 keV, c_2 = -120 keV, c_4 = 100 keV, v_0 = 71 keV and v_2 = -133 keV (Arima and Iachello, 1976a, p. 288). The broken lines denote calculated energies. The order of the levels corresponds to the scheme (10.22) and figure 10.4 respectively.

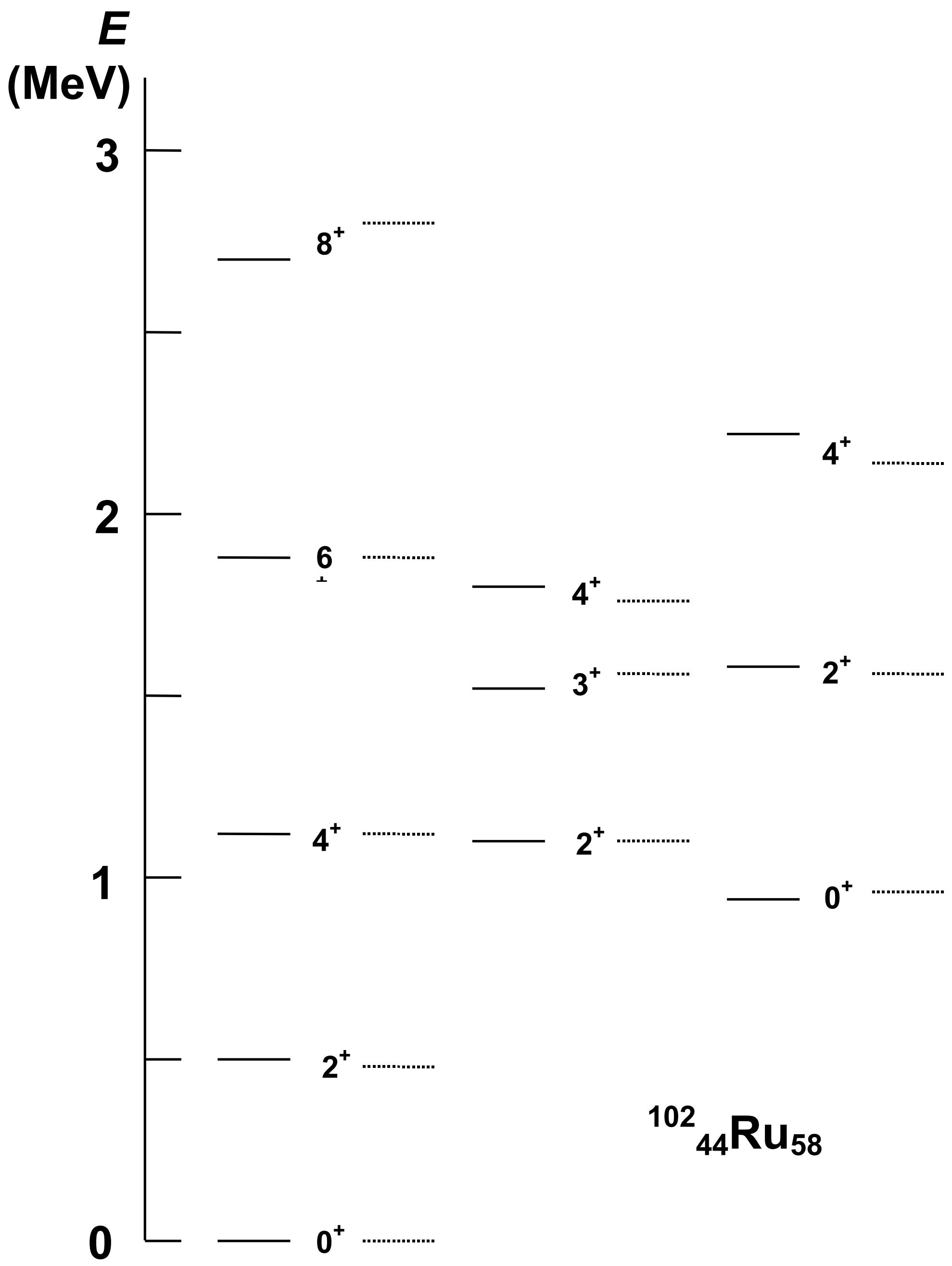

Figure 12.2. Comparison between the measured and the theoretical spectrum of ^{102}Ru. It is calculated with the parameters ε = 561 keV, c_0 = -177 keV, c_2 = -174 keV, c_4 = 81 keV, $v_2/\sqrt{2}$ = 78 keV, $v_0/2$ = - 86 keV (Kaup 1983, p. 16).The order of the levels corresponds to the scheme (10.22) and figure 10.3 respectively.

12.5 An empirical Hamilton operator

The authors Casten and Warner (1988, p. 440) describe a Hamilton operator with drastic simplifications compared with (12.1). They set $v_r = 0$ and $\boldsymbol{H}^{(\mathbf{I})} = \varepsilon\, \boldsymbol{n_d}$ i. e.

$$\boldsymbol{H} = \varepsilon\, \boldsymbol{n_d} + v_q\, \boldsymbol{Q}^2 \quad \text{with}$$

$$\boldsymbol{Q_\mu} = \boldsymbol{d}^+_\mu \boldsymbol{s} + \boldsymbol{s}^+ \boldsymbol{d}^\sim_\mu + \chi\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu\, . \tag{12.19}$$

The expression $\boldsymbol{Q_\mu}$ in (12.19) differs from (9.4) in the parameter χ. Similar simplifications as in (12.19) are usual in the interacting boson model 2 (chapter 15). Following the ideas of the IBM2, the expression (12.19) is made dependent on the number of proton-bosons, N_π, and on the number of neutron-bosons, N_ν. The conspicuous fact that nuclear properties depend largely on the product $N_\pi \cdot N_\nu$ (Casten, 1990, p. 235) is taken into account by writing the energy ε of a single d-boson like this

$$\varepsilon = \varepsilon_0 \cdot \mathrm{e}^{-\Theta\,(N_\pi \cdot N_\nu - N_0\,)}\, . \tag{12.20}$$

Properties of about 100 nuclei in three different regions of the nuclear card could be reproduced by varying weakly the parameters ε_0, Θ, N_0, v_q and χ

13 Lie algebras

In chapter 9 the importance of the Lie algebras for interpreting and evaluating the IBM-Hamiltonian has been stressed. In the present chapter the essential characteristics of these algebras will be outlined starting from well-known examples. In the first section the rules are described which hold for the elements of a Lie algebra. Looking into special matrices in sections 2 and 3 three classical, real Lie algebras are identified and their dimensions are put together in section 4. In the last section above all Casimir operators (chapter 9 and 14) will be treated.

13.1 Definition

A Lie algebra $\mathbb{L}$ comprises infinitely many elements *a, b, c, ...* which show common characteristics. For instance Lie algebras can be represented by matrices with certain symmetry properties or by operators. In particular the elements of a Lie algebra must be able to form Lie products or commutators $[a\,,\,b]$. For quadratic matrices a and b the well-known relation

$$[a\,,\,b] = ab - ba \tag{13.1}$$

holds, which results again in a matrix of the same type. If the elements of a Lie algebra are operators instead of matrices the commutator $[a\,,\,b]$ is still defined but the right hand side of (13.1) need not be satisfied.

The definition of a Lie algebra demands the following properties from its elements *a, b, c*

a) the commutator of two elements is again an element of the algebra

$$[a\,,\,b] \in \mathbb{L} \text{ for all } a, b \in \mathbb{L}, \tag{13.2}$$

b) a linear combination $\alpha\, a + \beta\, b$ of two elements with the numbers α and β is again an element of the algebra and the relation

$$[\alpha\, a + \beta\, b\,,\, c] = \alpha\,[a\,,\,c] + \beta\,[b\,,\,c] \tag{13.3}$$

holds (therefore the element 0 (zero) belongs also to the algebra),

c) interchanging both elements of a commutator results in

$$[a\,,\,b] = -\,[b\,,\,a], \tag{13.4}$$

d) finally “Iacobi’s identity“ has to be satisfied as follows

$$[a\,,\,[b\,,\,c]] + [b\,,\,[c\,,\,a]] + [c\,,\,[a\,,\,b]] = 0. \tag{13.5}$$

Anyhow, other kinds of coupling the elements are possible in principle. As an example the Casimir operators (13.32) can be mentioned, which result from multiplicative couplings of elements.

Furthermore, the definition demands that a Lie algebra has a finite dimension n i. e. it comprises a set of n elements $a_1, a_2, \ldots, a_n$ which act as a basis, by which every element x of the algebra can be represented like this

$$x = \sum_{i=1}^{n} \xi_i \, a_i \,. \tag{13.6}$$

In other words the algebra constitutes a n-dimensional vector space. The structure of the algebra can be formulated in terms of the basis. Because the commutator of two basis elements a_p and a_q belongs also to the algebra, according to (13.6) the relation

$$[a_p \, , \, a_q] = \sum_{r=1}^{n} c^{(r)}{}_{p\,q} \, a_r \tag{13.7}$$

holds. The numbers $c^{(r)}{}_{p\,q}$ are named structure constants. One formulates the commutator of the arbitrary elements x (see 13.6) and y ($= \sum_j \zeta_j \, a_j$) in terms of the structure constants like this

$$[x \, , \, y] = \sum_{i,\, j=1}^{n} \xi_i \, \zeta_j \, [a_i \, , \, a_j] = \sum_{i,\, j,\, r=1}^{n} \xi_i \, \zeta_j \, c^{(r)}{}_{i\,j} \, a_r \,. \tag{13.8}$$

If an algebra corresponds element by element to an other both are isomomorphic and we mark them with the same symbol. In the following we will restrict ourselves to real algebras which are characterised by real coefficients ξ_i in the linear combinations (13.6). With regard to the interacting boson model we will deal with the following types of real algebras: $u(N)$, $su(N)$ and $so(N)$.

13.2 The *u*(*N*)- and the *su*(*N*)-algebra

We maintain that quadratic and antihermitian matrices of the rank N ($N\times N$-matrices) form a Lie algebra. Antihermiticity means that the adjoint of the matrix a i. e. the conjugate complex and transposed (reflected on the diagonal) matrix $a^\dagger$ is identical with - a that is

$$a^\dagger = - a. \tag{13.9}$$

Providing real coefficients α and β, for two antihermitian matrices a and b with the same rank naturally the following relation holds

$$(\alpha \, a + \beta \, b)^\dagger = \alpha \, a^\dagger + \beta \, b^\dagger = - \alpha \, a - \beta \, b = - (\alpha \, a + \beta \, b). \tag{13.10}$$

Thus linear combinations of antihermitian matrices are again of this type. We make use of the well-known relation

$$(ab)^\dagger = b^\dagger \, a^\dagger \tag{13.11}$$

in order to look into the commutator of antihermitian matrices as follows

$$[a \, , \, b]^\dagger = (ab)^\dagger - (\mathrm{ba})^\dagger = b^\dagger a^\dagger - a^\dagger \, b^\dagger = ba - ab = - [a \, , \, b]. \tag{13.12}$$

Thus the commutator of two antihermitian matrices is again antihermitian.

Now it is not difficult to show that the conditions (13.3) up to (13.5) are met. This means that the antihermitian matrices of the rank N constitute a Lie algebra. It is named $u(N)$ algebra because it is closely related to the Lie group of the unitary matrices.

A matrix is antihermitian if every element of the matrix equals the negative, conjugate complex value of the element which is in the transposed position. Any $N\times N$-matrix of this kind can be composed linearly by the following quadratic matrices

$$\begin{pmatrix} i & 0 & . & & \\ 0 & 0 & . & & \\ . & . & . & & \\ & & & . & \\ & & & & . \end{pmatrix}, \begin{pmatrix} 0 & 0 & . & & \\ 0 & i & 0 & & \\ . & 0 & 0 & & \\ & & & . & \\ & & & & . \end{pmatrix}, \dots, \begin{pmatrix} 0 & i & 0 & & \\ i & 0 & . & & \\ 0 & . & . & & \\ & & & . & \\ & & & & . \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 0 & \\ 0 & 0 & 0 & & \\ i & 0 & . & & \\ 0 & & & . & \\ & & & & . \end{pmatrix},$$

$$\dots, \quad \begin{pmatrix} 0 & 1 & 0 & . & \\ -1 & 0 & & & \\ 0 & & . & & \\ & & & . & \\ & & & & . \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & & \\ -1 & 0 & . & & \\ 0 & & & . & \\ & & & & . \end{pmatrix}, \dots \quad . \tag{13.13}$$

Simple counting reveals that there are N^2 matrices in (13.13). As just mentioned they represent the basis of the Lie algebra $u(N)$. The number of basis elements is named dimension n. Thus for the Lie algebra $u(N)$ we have

$$n = N^2 . \tag{13.14}$$

Now we will investigate antihermitian matrices of the rank N with vanishing traces (sum of the diagonal elements). We expect that they constitute a Lie algebra as well. Obviously a linear combination of these matrices again has a vanishing trace. Furthermore, for quadratic matrices a and b the relation

$$trace\,(ab) = trace\,(ba). \tag{13.15}$$

holds. Thus the trace of a commutator vanishes always. This means that the conditions (13.2) up to (13.5) are satisfied and that we have a Lie algebra.

The trace of the basis matrices is always zero if in (13.13) all N diagonal matrices are replaced by matrices of the following form

$$\begin{pmatrix} i & 0 & . & & \\ 0 & -i & 0 & & \\ & 0 & . & & \\ & & & . & \\ & & & & . \end{pmatrix}, \begin{pmatrix} 0 & 0 & . & & \\ 0 & i & 0 & & \\ . & 0 & -i & 0 & \\ & & 0 & . & \\ & & & & . \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & . & & \\ 0 & . & i & & \\ & & & -i & \\ & & & & . \end{pmatrix}, \dots \quad . \tag{13.16}$$

Every diagonal, antihermitian matrix with trace 0 can be built by linearly combining matrices from (13.16). The elements (13.16) and the non-diagonal elements of (13.13) constitute the basis of the new Lie algebra. It comprises one matrix less than the one of the $u(N)$ algebra (13.13). Hence its dimension amounts to

$$n = N^2 - 1. \tag{13.17}$$

This algebra represents a special kind of the $u(N)$ algebra and is therefore named $su(N)$.

13.3 The *so*(*N*) algebra

The third Lie algebra, which we are investigating here, is constituted by real, antisymmetric and quadratic matrices of the rank N. The transposed a^{tr} of such a matrix a equals to $-a$

$$a^{tr} = -a\,.$$

Obviously each linear combination of these matrices is still antisymmetric. With the help of

$$(a\,b)^{tr} = b^{tr}\,a^{tr} \tag{13.18}$$

one shows in the same way as in the previous section that real, antisymmetric $N\times N$ matrices constitute a Lie algebra. As it is closely related to the Lie group of the orthogonal matrices it is named $o(N)$. Each diagonal element of an antisymmetric matrix is zero, i.e. its trace vanishes. Therefore its Lie algebra is named $so(N)$ as well in analogy to $su(N)$.

The basis elements of this algebra show the following structure

$$\begin{pmatrix} 0 & 1 & 0 & & \\ -1 & 0 & & & \\ 0 & 0 & . & & \\ & & & . & \\ & & & & . \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & & \\ -1 & 0 & . & & \\ & & & . & \\ & & & & . \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & & \\ 0 & 0 & 1 & & \\ 0 & -1 & 0 & & \\ & & & . & \\ & & & & . \end{pmatrix}, \dots\,. \tag{13.19}$$

and their number is $N(N-1)/2$. The dimension n of the Lie algebra $so(N)$ is thus

$$n = N(N-1)/2. \tag{13.20}$$

Specially for $N = 3$ the $so(3)$ basis elements read

$$a_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{13.21}$$

By calculating one obtains the following commutator relation

$[a_1 , a_2] = - a_3$ with cyclic permutations. (13.22)

The structure constants (13.7) of this algebra are therefore 1, 0 or -1. The minus sign in (13.22) disappears if one changes the sign of every basis matrix. The substitution

$e_1 = (a_1 + \mathrm{i}a_2)/\sqrt{2},\ \ e_{-1} = (a_1 - \mathrm{i}a_2)/\sqrt{2},\ \ e_0 = - \mathrm{i}a_3$ yields (13.23)

$[e_{-1} , e_1] = e_0,\ \ [e_{-1}, e_0] = e_{-1},\ \ [e_0 , e_1] = e_1$. (13.24)

Obviously the elements e_1, e_{-1} and e_0 constitute a Lie algebra with the dimension 3. It has the character of *so*(3) as can be shown. The commutation relations (13.24) agree completely with the rules (8.7) of the angular momentum operators J_μ . Thus they constitute the Lie algebra *so*(3).

13.4 Dimensions of three classical algebras

In table 13.1 the lowest dimensions *n* of the three Lie algebras mentioned above are put together. Group theory shows that there is only a small number of simple and real Lie algebras. Therefore, a simple algebra can be identified directly by means of the dimension. For the IBM1 the *n*-values of table 13.1 are sufficient. They facilitate to project a given Lie algebra on one of the algebras *u*(*N*), *su*(*N*) or *so*(*N*). The corresponding matrix algebra is a representation of the given Lie algebra. It is possible to transform the basis of one algebra (see 13.23 and 13.24) in order to obtain a corresponding algebra.

Table 13.1. The lowest dimensions n of the Lie algebras $u(N)$, $su(N)$ and $so(N)$ according to (13.14), (13.17) and (13.20).

Algebra :	$u(N)$	$su(N)$	$so(N)$	dimension n
Rank :	N	N	N	
		2	3	3
	2			4
			4	6
		3		8
	3			9
			5	10
		4	6	15
	4			16
			7	21
		5		24
	5			25
			8	28
		6		35
	6		9	36

13.5 Operators constituting Lie algebras, their basis functions and their Casimir operators

In the IBM Lie algebras constituted by operators prevail. A very important property of the operators is the existence of sets of basis functions $\psi^{(j)}{}_k$ which are affiliated to the operators. For example such basis functions are frequently represented by eigenstates of a physical system. They constitute a basis vector, the dimension of which can be chosen. Its elements can be combined linearly and generate a multidimensional vector space. They are characterised by ordering numbers.

In order to clarify the situation we consider the angular momentum operators $\boldsymbol{J_1}$, $\boldsymbol{J_{-1}}$ and $\boldsymbol{J_0}$ (8.6). They obey the commutation rules (8.7)

$$[\,\boldsymbol{J_{-1}}, \boldsymbol{J_0}\,] = \boldsymbol{J_{-1}}\,, \qquad [\,\boldsymbol{J_{-1}}, \boldsymbol{J_1}\,] = \boldsymbol{J_0}\,, \qquad [\,\boldsymbol{J_0}, \boldsymbol{J_1}\,] = \boldsymbol{J_1}\,. \qquad (13.25)$$

In section 13.3 we have seen that they constitute the Lie algebra $so(3)$. On the other hand, from quantum mechanics we know that the operators $\boldsymbol{J_1}$, $\boldsymbol{J_{-1}}$ and $\boldsymbol{J_0}$ generate functions or quantum states $\varphi_{j\,m}$, which are characterised by the quantity j (an integer for bosons). For a given j there are $2j + 1$ functions, which are labelled by m (projection of the spin or of the angular momentum). If an operator $\boldsymbol{J_i}$ acts on $\varphi_{j\,m}$ the result is not an entirely new function but as we know from quantum mechanics the relations

$$J_1\varphi_{j\,m} = -\,2^{-1/2}\,[j(j+1) - m(m+1)]^{1/2}\,\varphi_{j,\,m\,+1},$$

$$J_{-1}\varphi_{j\,m} = \;\;2^{-1/2}\,[j(j+1) - m(m-1)]^{1/2}\,\varphi_{j,\,m\,-1}\,,$$

$$J_0\varphi_{j\,m} \;= m\varphi_{j\,m} \tag{13.26}$$

hold. Thus when acting on a function $\varphi_{j\,m}$ the operators J_1, J_{-1} or J_0 generate again a function of this kind with the same j. In general an operator of a Lie algebra creates a linear combination of basis functions when it acts on such a function.

We turn to the operator

$$J^2 = J_x^{\,2} + J_y^{\,2} + J_z^{\,2} = -J_1\,J_{-1} - J_{-1}\,J_1 + J_0^{\,2} \tag{13.27}$$

defined in (8.8). With the aid of (13.25) one shows that the following relation holds

$$[J^2\,,\,J_i] = 0 \text{ for } i = -\,1,\,0,\,1. \tag{13.28}$$

We see that the operator J^2, which does not to belong to the Lie algebra (J_1, J_{-1}, J_0), commutes with the basis elements and therefore with all elements of the algebra. Operators which behave like J^2 are named **Casimir operatores**. In physics they are important because they generate eigenvalues. In the case of J^2 we have according to (8.5)

$$J^2\varphi_{j\,m} = j(j+1)\;\varphi_{j\,m}\,. \tag{13.29}$$

Thus, when the Casimir operator J^2 acts on the basis function $\varphi_{j\,m}$, this is reproduced and supplied with a factor, which depends only on j.

Now we can put together the properties of the Lie algebras, which we need for the IBM. We refer to the situation of the angular momentum and abstain from group theoretical proofs.

13.6 Properties of operators

We start with basis operators $\Phi^{(i)}$ with i = 1, .. ,n which are linearly independent, constitute a Lie algebra according to (13.2) up to (13.5) and are connected by the structure constants $c^{(r)}{}_{p\,q}$ (13.7). There exist sets of basis functions $\psi^{(j)}{}_k$ (j stands for the set) on which the operators $\Phi^{(i)}$ act like this

$$\Phi^{(i)}\,\psi^{(j)}{}_k = \sum_{l\,=\,1}{}^{n}\,\Gamma^{(i,j)}{}_{k\,l}\,\psi^{(j)}{}_l \tag{13.30}$$

analogously to (13.26). The Casimir operators Ξ, which commute with all operators $\Phi^{(i)}$ according to (13.28) as follows

$$[\Xi\;,\,\Phi^{(i)}] = 0\,, \tag{13.31}$$

are formed in terms of the elements $\Phi^{(i)}$. We need the linear type Ξ_1 of that operator and the quadratic Ξ_2. The last has the following form

$$\Xi_2 = \sum_{i,j}{}^{n} X_{ij}\, \boldsymbol{\Phi}^{(i)}\, \boldsymbol{\Phi}^{(j)}. \qquad (13.32)$$

The coefficients X_{ij} of the quadratic Casimir operator depend from the structure constants $c^{(r)}{}_{p\,q}$ (13.7). For example the coefficients of the Casimir operator $\boldsymbol{J}^2$ read according to (13.27) : $X_{1,-1} = X_{-1,1} = -1$, $X_{00} = 1$ and all other vanish.

The sets of functions $\psi^{(j)}{}_k$, which have so-called irreducible matrices $\Gamma^{(i,j)}{}_{k\,l}$ (13.30), generate eigenvalues $C^{(j)}{}_2$ under the influence of the Casimir operator Ξ_2 like this

$$\Xi_2\, \psi^{(j)}{}_k = C^{(j)}{}_2\, \psi^{(j)}{}_k. \qquad (13.33)$$

According to (13.29) the eigenvalue of the quadratic Casimir operator $\boldsymbol{J}^2$ of the Lie algebra *so*(3) has the form $j(j + 1)$.

Using creation and annihilation operators for bosons, Frank and Van Isacker (1994, p. 310) show that the eigenvalues of the Casimir operators of the *so*(N)-algebras have the following form

$$C^{(\rho)}{}_{2,\, so(N)} = \rho\,(\rho + N - 2). \qquad (13.34)$$

The quantity N is the rank of the *so*-algebra and ρ is an integer which is limited by a function of the boson number. For instance for $N = 3$ (13.34) agrees with (13.29), provided that ρ is replaced by the angular momentum J of the state. We know that in the *u*(5)-limit of the IBM the quantity J has the values $2n_d$, $2n_d - 2$, $2n_d - 3$, ... , 1, 0. In the next chapter we will meet further applications of formula (13.34).

For the Lie algebra *su*(3) Cornwell (1990, p. 596) derives the following eigenvalue of the quadratic Casimir operator (apart from a factor 1/9)

$$C^{(n,\,m)}{}_{2,su(3)} = n^2 + m^2 + nm + 3n + 3m. \qquad (13.35)$$

The integers n and m characterise the irreducible matrices ($\Gamma^{(i,\,n\,m)}{}_{k\,l}$) (13.30) of the algebra and the affiliated functions $\psi^{(n\,m)}{}_l$ .The expression (13.35) will arise in the next chapter in connection with the second special case of the IBM1.

The basis operators of the IBM (chapter 14) not only constitute a Lie algebra in all but they contain subsets which are subalgebras. This means for instance that commutators of operators of a subalgebra are linear combinations of elements of this subset (13.2). The Casimir operator of the higher algebra commutes with the Casimir operator of the subalgebra, which follows from $\Xi\, \boldsymbol{\Phi}^{(i)} = \boldsymbol{\Phi}^{(i)}\, \Xi$ (13.31) and from (13.32). The Casimir operators of the subalgebras, which we will meet in the next chapter, are summands in the Casimir operators of the higher algebra.

Furthermore, it is true that the basis function $\psi^{(j)}{}_k$ (13.30) of the higher algebra is also the basis function of the subalgebra. Therefore this function is labelled both with the index of the Casimir eigenvalue C_2 of the higher algebra and with the corresponding index of the subalgebra. For example in the basis function

φ_{jm} (13.29) the quantity m is the characteristic parameter of the subalgebra *so*(2) (consisting only of the operator $\boldsymbol{J_z} = \boldsymbol{J_0}$) and j is the index of the higher algebra *so*(3), which comprises $\boldsymbol{J_{-1}}$, $\boldsymbol{J_1}$ and $\boldsymbol{J_0}$.

14 Group theoretical aspects of the IBM1

In this chapter first we will show that the IBM-Hamiltonian is composed of elements which constitute a Lie algebra. In the second section the affiliated subalgebras are looked into. Three chains of subalgebras can be formulated. The third section is rather illustrative and shows how the basis operators of the IBM act on the accompanying basis functions. In sections 4, 5 and 6 special cases of the IBM are treated. The eigenvalues of the Casimir operators which belong to these cases, determine the energy eigenvalues. Comparisons with measured level schemes will be performed.

14.1 Basis operators of the Lie algebras in the IBM1

In this section we show first that the Hamilton operator of the IBM1 is composed exclusively of the following operators

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}{}_M\,,\ [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_M\,,\ [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}{}_M \text{ and } [\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}{}_0 \qquad (14.1)$$

Owing to (9.37) we can write the Hamiltonian like this

$$\boldsymbol{H} = \varepsilon_n \boldsymbol{N} + \varepsilon_d{}' \boldsymbol{n}_d + v_n \boldsymbol{N}^2 + v_{nd} \boldsymbol{N}\boldsymbol{n}_d + v_d \boldsymbol{n}_d^2 + v_r \boldsymbol{R}^2 + v_t \boldsymbol{T}^2 + v_j \boldsymbol{J}^2 + v_q \boldsymbol{Q}^2. \qquad (14.2)$$

The cited operators - apart from $\boldsymbol{R}^2$ and $\boldsymbol{T}^2$ - have the following structure (9.1), (9.4)

$$\boldsymbol{n}_d = \sqrt{5}\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}{}_0,$$

$$\boldsymbol{N} = \boldsymbol{s}^+\boldsymbol{s} + \boldsymbol{n}_d = [\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}{}_0 + \sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}{}_0,$$

$$\boldsymbol{J}^2 = -\sqrt{3}\cdot 10\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}\,]^{(0)}{}_0 =$$

$$10 \sum_\mu (-1)^\mu\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}{}_\mu\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}{}_{-\mu}\,, \qquad (14.3)$$

$$\boldsymbol{Q}^2 = \sum_\mu (-1)^\mu\, \boldsymbol{Q}_\mu \boldsymbol{Q}_{-\mu}$$

with $\quad \boldsymbol{Q}_\mu = [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu + [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}{}_\mu - (\sqrt{7}/2)\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}{}_\mu\,.$

Thus these operators are built by elements of the type (14.1). This is also true for both remaining operators in (14.2), which we will show by remodelling. We write the seniority operator $\boldsymbol{T}^2 = \boldsymbol{n}_d\,(\boldsymbol{n}_d + 3) - 5\cdot[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}{}_0\, [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)}{}_0$ (9.2) with the aid of (9.12) this way

$$\boldsymbol{T}^2 = \boldsymbol{n}_d^2 + 4\boldsymbol{n}_d - \sum_{J'} \sqrt{(2J' + 1)}\cdot[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)}\,.$$

We replace $\boldsymbol{n}_d^2$ making use of (9.20)

$$\boldsymbol{n}_d^2 = 5[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}]^{(0)} = \qquad (14.4)$$

$$\sum_{J''} (-1)^{J''} \sqrt{(2J'' + 1)}[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J'')}]^{(0)} - 4\, \boldsymbol{n}_d$$

and obtain

$$T^2 = -2 \sum_{J'=1,3} \sqrt{(2J'+1)} \cdot [[d^+ \times d^\sim]^{(J')} \times [d^+ \times d^\sim]^{(J')}]^{(0)} . \qquad (14.5)$$

The remaining operator R^2 (9.3) reads

$$R^2 = N(N+4) - (\sqrt{5} \cdot [d^+ \times d^+]^{(0)} - s^+s^+)(\sqrt{5}[d^\sim \times d^\sim]^{(0)} - ss) =$$

$$N(N+4) - 5 \cdot [[d^+ \times d^+]^{(0)} \times [d^\sim \times d^\sim]^{(0)}]^{(0)} + \sqrt{5}[d^+ \times d^+]^{(0)}\, ss +$$

$$\sqrt{5}\, s^+s^+[d^\sim \times d^\sim]^{(0)} - s^+s^+\, ss . \qquad (14.6)$$

With the help of (9.2) we obtain

$$R^2 = N(N+4) - T^2 - n_d(n_d+3) + \sqrt{5}[d^+ \times d^+]^{(0)}\, ss +$$

$$\sqrt{5}\, s^+s^+[d^\sim \times d^\sim]^{(0)} - s^+s^+\, ss,$$

insert T^2 from (14.5) and transform like this

$$R^2 = N(N+4) - n_d(n_d+3) - 2 \sum_{J'=1,3} \sqrt{(2J'+1)} \cdot [[d^+ \times d^\sim]^{(J')} \times [d^+ \times d^\sim]^{(J')}]^{(0)} +$$

$$\sqrt{5}[[d^+ \times s]^{(2)} \times [d^+ \times s]^{(2)}]^{(0)} + \sqrt{5}[[s^+ \times d^\sim]^{(2)} \times [s^+ \times d^\sim]^{(2)}]^{(0)} -$$

$$[[s^+ \times s]^{(0)} \times [s^+ \times s]^{(0)}]^{(0)} + [s^+ \times s]^{(0)}. \qquad (14.7)$$

Thus, the entire Hamilton operator is built by operators of the type (14.1).

Now we show that the operators (14.1) constitute a Lie algebra. For the moment we are restricting ourselves to operators such as $[d^+ \times d^\sim]^{(J)}{}_M$. According to (A8.4) the commutation relation

$$[[d^+ \times d^\sim]^{(J')}{}_{M'} , [d^+ \times d^\sim]^{(J'')}{}_{M''}] =$$

$$\sum_{k\,\kappa} [d^+ \times d^\sim]^{(k)}{}_{\kappa} \cdot \sqrt{(2J'+1)}\sqrt{(2J''+1)} \cdot \qquad (14.8)$$

$$(J'\, M'\, J''\, M'' \mid k\, \kappa) \left\{ \begin{matrix} J' & J'' & k \\ 2 & 2 & 2 \end{matrix} \right\} (-1)^k (1 - (-1)^{J'+J''+k})$$

holds. That is, the commutator of two operators $[d^+ \times d^\sim]^{(J)}{}_M$ is a linear combination of such operators. Therefore the defining conditions (13.2) and (13.7) for Lie algebras are satisfied. Interchanging both operators in the commutator (14.8) results in altering the sign of the Clebsch-Gordan coefficient (consult (A1.4) and the relation $J' + J'' + k$ = odd). Consequently the condition (13.4) is met and one can show also that the relations (13.3) and (13.5) are satisfied. Therefore the operators of the type $[d^+ \times d^\sim]^{(J)}{}_M$ constitute a Lie algebra.

The operators in (14.1) which contain ***s***-operators satisfy the following commutation relations

$$[[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0 , [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu] = [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu ,$$

$$[[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu , [\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0] = [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu ,$$

$$[[\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\kappa , [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu] = [[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\kappa , [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu] = 0,$$

$$[[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\kappa , [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu] = \tag{14.9}$$

$$\sum_{J\,M} (2\;\kappa\;2\;\mu \mid J\,M)\,[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_M - \delta_{\kappa,-\mu}(-1)^\mu\,[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0 ,$$

$$[[\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu , [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_M] = (2, -\mu, 2, \mu + M \mid J\,M)\,(-1)^\mu\,[\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_{\mu+M} ,$$

$$[[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu , [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_M] = -\,(2, \mu + M, 2, -\mu \mid J\,M)\,(-1)^\mu\,[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_{\mu+M}$$

which is shown similarly as in (A8.4). From (14.8) and (14.9) follows that all operators listed in (14.1) constitute a Lie algebra.

14.2 Subalgebras within the IBM1

We now will investigate how this algebra can be subdivided in subalgebras. We start with the elements $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}_\mu$. According to (8.9) these operators represent the ladder operators $\boldsymbol{J}_\mu$ of the angular momentum apart from the factor $\sqrt{10}$. Owing to (13.24) they constitute the Lie algebra *so*(3).

We turn to elements of the type

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_\mu \text{ with } J = 1, 3. \tag{14.10}$$

According to (14.8) the commutator of two elements of this kind creates a linear combination of operators $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_\mu$ (with J = 1, 3). Obviously they also meet the other definition conditions (section 13.1) and constitute a Lie algebra of the dimension (= number of basis elements) n = 3 + 7 = 10. Table 13.1 contains only one Lie algebra with n = 10 namely the algebra *so*(5).

The totality of the elements $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_M$ with $0 \le J \le 4$ also makes up a Lie algebra. By means of its dimension n = 1 + 3 + 5 + 6 + 9 = 25 we learn from table 13.1 that it is the algebra *u*(5).

If one adds the elements $[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_M$, $[\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_M$ and $[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0$, the resulting 36 elements constitute a Lie algebra, which has been mentioned in the last section. Table 13.1 reveals that it is the algebra *u*(6).

Consequently the basis elements (14.1) constitute the following chain of Lie algebras

$$\text{I.} \qquad u(6) \supset u(5) \supset so(5) \supset so(3). \tag{14.11}$$

It defines the vibrational or *u*(5) limit (chapter 10), which will be reviewed in section 14.4 once more.

Now we are claiming that the elements

$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_\kappa$ and $\boldsymbol{Q}_\mu = [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu + [\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu - (\sqrt{7}/2)\, [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu$ (14.12)

make up a Lie algebra. First we investigate the commutator

$$[\boldsymbol{Q}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_\kappa] = [[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_\kappa] + \tag{14.13}$$
$$[[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_\kappa] - (\sqrt{7}/2)\, [[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_\kappa].$$

With the aid of (14.8), (14.9), (A1.4) up to (A1.6) and (A3.14) one obtains finally

$$[\boldsymbol{Q}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_\kappa] = (-1)^\mu\, (2, -\mu, 2, \mu + \kappa \,|\, 1\ \kappa)\, \boldsymbol{Q}_{\mu+\kappa}\,. \tag{14.14}$$

The Clebsch-Gordan coefficient prevents that $|\mu + \kappa|$ exceeds 2. Furthermore, we need the commutator $[\boldsymbol{Q}_\mu , \boldsymbol{Q}_\kappa]$, for which we write the non vanishing terms

$$[\boldsymbol{Q}_\mu , \boldsymbol{Q}_\kappa] = [[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu , [\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\kappa] - [[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\kappa , [\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu] -$$
$$(\sqrt{7}/2)\, [[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\kappa] + (\sqrt{7}/2)\, [[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\kappa , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu] -$$
$$(\sqrt{7}/2)\, [[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\kappa] + (\sqrt{7}/2)\, [[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\kappa , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu] +$$
$$(7/4)\, [[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\kappa]. \tag{14.15}$$

The four terms with the factor $\sqrt{7}/2$ in (14.15) cancel out in pairs because of (14.9), (A1.4) and (A1.6). By means of the explicit values of the 6-*j* symbols (A3.14)

$$\begin{Bmatrix} 2 & 2 & 1 \\ 2 & 2 & 2 \end{Bmatrix} = -\,1/10, \quad \begin{Bmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \end{Bmatrix} = 4/35$$

and of (14.8) and (14.9) one obtains the following result

$$[\boldsymbol{Q}_\mu , \boldsymbol{Q}_\kappa] = (15/4)\, (2\ \mu\ 2\ \kappa \,|\, 1, \kappa + \mu)\, [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_{\kappa+\mu}\,. \tag{14.16}$$

The Clebsch-Gordan coefficient prevents that $|\mu + \kappa|$ exceeds 1. In this way the choice of the factor $\sqrt{7}/2$ is explained. It causes the term $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}{}_M$ in (14.16) to vanish. Clearly the factor $\sqrt{7}/2$ might have also a positive sign in (14.12) because in (14.15) it appears only in squared form.

Thus the operators (14.12) constitute a Lie algebra with the dimension $n = 3 + 5 = 8$, which is in fact the *su*(3) algebra according to table 13.1. It includes the elements $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_\kappa$, which make up the algebra *so*(3), as we know.

If we add the operators $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(k)}{}_\mu$ ($k = 0, 3,4$), $[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu$, $[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu$ and $[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}{}_0$ to the 8 elements of (14.12) all possible commutators create linear combinations of these 36 elements. In case single elements $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu$ are generated, they have to be replaced by $([\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}{}_\mu + [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu - \boldsymbol{Q}_\mu)\, 2/\sqrt{7}$. The totality constitutes the algebra *u*(6), which is subdivided in subalgebras like this

$$\text{II.} \qquad u(6) \supset su(3) \supset so(3). \tag{14.17}$$

This chain represents the $su(3)$ limit, which is named rotational limit as well. In section 14.5 it will be looked into.

We come to the third special case of the IBM by the following set of elements

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_\mu \; (J = 1, 3) \text{ and } \boldsymbol{S}_\lambda = [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\lambda + [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\lambda . \qquad (14.18)$$

The first group of operators in (14.18) makes up the algebra $so(5)$ as we know from (14.11). According to (14.9) the relation

$$[\boldsymbol{S}_\lambda , \boldsymbol{S}_\mu] = 2 \sum_{J = 1,3} (2\, \lambda\, 2\, \mu \mid J, \lambda + \mu)\, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_{\lambda + \mu} \qquad (14.19)$$

holds and for odd J we have

$$[\boldsymbol{S}_\lambda , [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_\mu] = \qquad (14.20)$$

$$- ((2J + 1)/5)^{1/2} (2\, \lambda\, J\, \mu \mid 2, \lambda + \mu) ([\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_{\lambda+\mu} + [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_{\lambda+\mu}).$$

Therefore, the elements of (14.18) constitute a Lie algebra of the dimension n = 3 + 7 + 5 + = 15. In table 13.1 the algebras $su(4)$ and $so(6)$ have this dimension. They are isomorphic as can be shown.

Adding the elements $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}_\mu$ (J = 0, 3, 4), $[\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}_\mu$ and $[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0$ to the set (14.18) one obtains 36 elements constituting the $u(6)$ algebra, which is shown analogously to the derivation of (14.17). Thus we obtain the following chain of algebras

$$\text{III.} \quad u(6) \supset so(6) \supset so(5) \supset so(3). \qquad (14.21)$$

This is the third special case of the IBM, which is named $so(6)$ or γ -instable limit.

In table 14.1 the chains of algebras of the IBM1 and the affiliated basis elements are put together.

Table 14.1. Chains of algebras for the three special cases of the IBM1

Basis elements	dimension n	algebra	
$[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu$, $[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu$, $[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0$	36	$u(6)$	
		$\cup$	
$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(0)}_0$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(4)}_\mu$	25	$\underline{u(5)}$	
		$\cup$	vibrational
$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}_\mu$	10	$so(5)$	limit
		$\cup$	
$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}_\mu$	3	$so(3)$	
$[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu$, $[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu$, $[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(0)}_0$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}_\mu$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(4)}_\mu$	36	$u(6)$	
		$\cup$	
$[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu + [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu - (\sqrt{7}/2)[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu$	8	$\underline{su(3)}$	rotational
		$\cup$	limit
$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}_\mu$	3	$so(3)$	
$[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu$, $[\boldsymbol{s}^+ \times \boldsymbol{s}]^{(0)}_0$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(0)}_0$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu$, $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(4)}_\mu$	36	$u(6)$	
		$\cup$	
$[\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu + [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}_\mu$	15	$\underline{so(6)}$	
		$\cup$	γ-instable
$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}_\mu$	10	$so(5)$	limit
		$\cup$	
$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}_\mu$	3	$so(3)$	

14.3 The spherical basis as function basis of Lie algebras

In chapter 6 we have treated the seniority scheme and represented its boson states like this (6.18)

$$| n_s\, n_d\, \tau\, n_\Delta\, J\, M \rangle =$$

$$((2\tau + 3)!!/(n_s!(n_d + \tau + 3)!!(n_d - \tau)!!))^{1/2} (\boldsymbol{s}^+)^{n_s} (\boldsymbol{P}^+)^{(n_d - \tau)/2}\, | \tau\, \tau\, n_\Delta\, J\, M \rangle \qquad (14.22)$$

with $\boldsymbol{P}^+ = \sqrt{5}\, [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}_0$ (6.12).

These so-called spherical basis states are assigned to the $u(5)$- or vibrational limit (chapter 10) and we know that they are eigenfunctions of the operators $\boldsymbol{J}^2$ and $\boldsymbol{T}^2$, which are components of the corresponding Hamiltonian. As an exercise we check if these states behave like basis functions of the $u(5)$ limit and if they satisfy the relation (13.30). We restrict ourselves to the $so(5)$ subalgebra. It comprises the elements

$$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(1)}{}_{\kappa} \text{ and } [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}{}_{\nu} \,. \tag{14.23}$$

According to (13.30) we expect the following behaviour: when an operator of the type (14.23) acts on a basis function (14.22) a linear combination of such functions arises.

The equations (13.26) show that the first operator in (14.23) satisfies this condition because it agrees with the angular momentum operator $\boldsymbol{J}_{\mu}$ (8.9) apart from a factor.

Now we let the operator $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}{}_{\kappa}$ act on the state (14.22) and move it to the right in front of the operator $\boldsymbol{P}^+$. Then we maintain that these operators commute. In order to show it, according to (A9.3) and (A1.16) we can write for odd k

$$[[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(k)}{}_{\kappa} , [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}{}_0] = \tag{14.24}$$

$$2 \sum_{\mu} (2\ \mu\ 2, -\mu \mid 0\ 0)\ (2, \kappa + \mu, 2, -\mu \mid k\ \kappa)\ (-1)^{-\mu} \boldsymbol{d}^+_{\kappa+\mu} \boldsymbol{d}^+_{-\mu} =$$

$$(2/\sqrt{5}) \sum_{\mu} (2, \kappa + \mu, 2, -\mu \mid k\ \kappa)\ \boldsymbol{d}^+_{\kappa+\mu} \boldsymbol{d}^+_{-\mu} = (2/\sqrt{5})\ [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(k)}{}_{\kappa} \,.$$

Because k is odd the last expression vanishes owing to (6.6). Therefore $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}{}_{\kappa}$ (k = 1, 3) can be interchanged with $\mathbf{P}^+$. Generally the relation

$$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(k)}{}_{\kappa} (\boldsymbol{P}^+)^{(n_d - \tau)/2} = (\boldsymbol{P}^+)^{(n_d - \tau)/2} [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(k)}{}_{\kappa} , \; k = 1, 3, .. \,, \tag{14.25}$$

holds. This means that the $so(5)$ operator (14.23) can be moved in front of the state $| \, n_s\ \tau\ \tau\ n_{\Delta}\ J\ M \rangle$ with maximal seniority in (14.22). For the time being we are restricting ourselves to states with $n_d = \tau = 2$. They read according to (6.5)

$$| \, n_d = \tau = 2, J\, M \rangle = (2)^{-1/2} [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)}{}_M | \rangle \text{ with } J = 2, 4. \tag{14.26}$$

Acting on the state (14.26) the $so(5)$ operator $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}{}_{\kappa}$ generates the expression

$$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}{}_{\kappa} (2)^{-1/2} [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)}{}_M | \rangle = \tag{14.27}$$

$$\sqrt{2} \sum_{\mu} (2\ \mu\ 2, M - \mu \mid J\ M)\ (2, \kappa + \mu, 2, -\mu \mid 3\ \kappa)\ (-1)^{\mu}\ \boldsymbol{d}^+_{\kappa+\mu} \boldsymbol{d}^+_{M-\mu} \,,$$

which is shown by means of (9.3) and the relation $\boldsymbol{d}_{\mu} | \rangle = 0$. With the aid of (A1.14) the pair of $\boldsymbol{d}^+$-operators is transformed to expressions such as $[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J')}{}_{M'}$. Because we are interested in the question if $J' = 0$ arises, we investigate situations with $M' = 0$. That is we put $M = -\kappa$ in (14.27) and obtain

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(3)}{}_\kappa\, (2)^{-1/2}[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)}{}_{-\kappa}\,|\rangle =$$

$$\sqrt{2}\,\sum_\mu (2\ \mu\ 2, -\kappa - \mu \mid J, -\kappa)\ (2, \kappa + \mu, 2, -\mu \mid 3\ \kappa)\ (-1)^\mu\ \boldsymbol{d}^+_{\kappa+\mu}\boldsymbol{d}^+_{-\kappa-\mu} =$$

$$\sqrt{2}\,\sum_\mu (2\ \mu\ 2, -\kappa - \mu \mid J, -\kappa)\ (2, \kappa + \mu, 2, -\mu \mid 3\ \kappa)\ (-1)^\mu \cdot$$

$$\sum_{J'} (2, \kappa + \mu, 2, -\kappa - \mu \mid J'\,0)\ [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J')}{}_0\,|\rangle. \qquad (14.28)$$

With the help of (A1.16), (A1.4) and (A1.5) we write the coefficient for $J' = 0$ as follows

$$\sqrt{(2/5)}\ (-1)^\kappa \sum_\mu (2\ \mu\ 2, -\kappa - \mu \mid J, -\kappa)\ (2\ \mu\ 2, -\kappa - \mu \mid 3, -\kappa) = 0. \qquad (14.29)$$

Owing to the orthogonality relation (A1.10) this expression vanishes because J has only the values 2 or 4 (14.26).

Thus we have shown that the operator $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(3)}{}_\kappa$ acting on the state (14.26) which has maximal seniority creates a linear combination of states of the same kind. In other words, when the operator $[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(3)}{}_\kappa$ acts on a state with seniority 2, the resulting states have the same seniority. That is, the rule (13.30) for Lie algebras is satisfied.

It's obvious that this statement holds also for states with $\tau = 1$, i. e. $\boldsymbol{d}^+|\rangle$. For $\tau = 3$ it can be checked with the aid of (6.9). The proof of the general relation

$$[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(3)}{}_\kappa \mid N\, n_d\, \tau\, n_\Delta\, J\, M \rangle =$$

$$\sum_{n'_\Delta J' M'} \Gamma^{(3,\,\kappa)}{}_{N\, n_d\, \tau\, n_\Delta\, J\, M\, n'_\Delta J' M'} \mid N\, n_d\, \tau\, n'_\Delta\, J'\, M' \rangle \qquad (14.30)$$

is laborious.

14.4 Casimir operators of the *u*(5)- or vibrational limit

In chapter 9 the Hamilton operator was transformed in a combination of operators which we named Casimir operators. In sections 14.4 up to 14.6 we will show now that this naming is correct. We begin with the special form (10.1) of the Hamiltonian characterising the *u*(5) limit

$$\boldsymbol{H}^{(\mathrm{I})} = \varepsilon_n \boldsymbol{N} + v_n \boldsymbol{N}^2 + (\varepsilon_d{}' + v_{nd} \boldsymbol{N})\boldsymbol{n_d} + v_d\, \boldsymbol{n_d}^2 + v_t \boldsymbol{T}^2 + v_j \boldsymbol{J}^2. \qquad (14.31)$$

We will investigate the properties of the operators in (14.31), confront it with the rules (13.31) up to (13.33) and make visible how they are related to the subalgebras of the *u*(5) limit.

According to (13.28) the operator $\boldsymbol{J}^2 = -10\sqrt{3}\cdot[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}]^{(0)}$ (8.14) commutes with the components $\boldsymbol{J}_\mu = \sqrt{10}\ [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}{}_\mu$, $\mu = 1, 0, -1$ (8.9), which constitute the Lie algebra *so*(3) (13.24). Clearly $\boldsymbol{J}^2$ must be its quadratic Casimir operator. It satisfies not only (13.31) and (13.32) but it generates also the eigenvalue $J(J + 1)$ and therefore it meets the conditions (13.34).

We expect that

$$T^2 = -2 \sum_{J=1,3} \sqrt{(2J+1)}\cdot[[d^+ \times d^\sim]^{(J)} \times [d^+ \times d^\sim]^{(J)}]^{(0)} =$$

$$2 \sum_{J=1,3} \sum_M (-1)^M [d^+ \times d^\sim]^{(J)}{}_M [d^+ \times d^\sim]^{(J)}{}_{-M}$$

is the quadratic Casimir operator of the higher algebra *so*(5). Firstly it is composed of products of operators $[d^+ \times d^\sim]^{(k)}{}_\kappa$, $k = 1, 3$ of this algebra and obeys therefore the relation (13.32). According to (10.13) T^2 meets the condition (13.33) and creates the eigenvalue $\tau(\tau + 3)$, which has the form (13.34), because the rank N amounts to 5. According to (10.13) and (14.30) we have

$$[d^+ \times d^\sim]^{(3)}{}_\kappa T^2 \,|\, N\, n_d\, \tau\, n_\Delta\, J\, M\rangle = \tau(\tau + 3)\, [d^+ \times d^\sim]^{(3)}{}_\kappa |N\, n_d\, \tau\, n_\Delta\, J\, M\rangle =$$

$$\tau(\tau + 3) \sum_{n'_\Delta J' M'} \Gamma^{(3,\kappa)}{}_{N\, n_d\, \tau\, n_\Delta\, J\, M\, n'_\Delta J' M'} \,|\, N\, n_d\, \tau\, n'_\Delta\, J'\, M'\rangle =$$

$$T^2 \sum_{n'_\Delta J' M'} \Gamma^{(3,\kappa)}{}_{N\, n_d\, \tau\, n_\Delta\, J\, M\, n'_\Delta J' M'} \,|\, N\, n_d\, \tau\, n'_\Delta\, J'\, M'\rangle =$$

$$T^2 [d^+ \times d^\sim]^{(3)}{}_\kappa \,|\, N\, n_d\, \tau\, n_\Delta\, J\, M\rangle. \tag{14.32}$$

Thus the condition (13.31) is met, which is valid also for the basis operator $[d^+ \times d^\sim]^{(1)}{}_\kappa$.

The next higher algebra *u*(5) in this special case of the IBM1 comprises all elements $[d^+ \times d^\sim]^{(k)}{}_\kappa$, $k = 0$ up to 4. According to (14.4) the relation

$$n_d^2 + 4\, n_d = \sum_{k=0}^{4} (-1)^k \sqrt{(2k+1)}[[d^+ \times d^\sim]^{(k)} \times [d^+ \times d^\sim]^{(k)}]^{(0)} =$$

$$\sum_{k=0}^{4} \sum_\kappa (-1)^\kappa [d^+ \times d^\sim]^{(k)}{}_\kappa [d^+ \times d^\sim]^{(k)}{}_{-\kappa} \tag{14.33}$$

holds. This operator must be the quadratic Casimir operator of the algebra *u*(5) because it satisfies (13.31) up to (13.33). We guess that the number operator $n_d = \sum_\mu (-1)^\mu d^+_\mu d^\sim_{-\mu}$ is the linear Casimir operator of the *u*(5) algebra.

The highest algebra in the *u*(5) chain, *u*(6), comprises all 36 elements of the type $[b^+_l \times b^\sim_{l'}]^{(k)}{}_\kappa$ ($l, l' = 0, 2$; $b_0 = s$, $b_{2,\mu} = d_\mu$). We maintain that its quadratic Casimir operator reads as follows in agreement with (13.32)

$$C_{u(6)} = \sum_{l, l', l'', l''' = 0,2} \sum_{k\,\kappa} (-1)^\kappa [b^+_l \times b^\sim_{l'}]^{(k)}{}_\kappa [b^+_{l''} \times b^\sim_{l'''}]^{(k)}{}_{-\kappa} \quad \text{i. e.} \tag{14.34}$$

$$C_{u(6)} = \sum_{k=0}^{4} \sum_\kappa (-1)^\kappa [d^+ \times d^\sim]^{(k)}{}_\kappa [d^+ \times d^\sim]^{(k)}{}_{-\kappa} +$$

$$\sum_\kappa (-1)^\kappa ([d^+ \times s]^{(2)}{}_\kappa [s^+ \times d^\sim]^{(2)}{}_{-\kappa} + [s^+ \times d^\sim]^{(2)}{}_\kappa [d^+ \times s]^{(2)}{}_{-\kappa}) +$$

$$[s^+, s]^{(0)} [s^+, s]^{(0)}. \tag{14.35}$$

With the aid of (14.33), (6.35), (6.36), (5.36) and little conversions one obtains

$$C_{u(6)} = n_d^2 + 4\, n_d + n_d(1 + n_s) + n_s(n_d + 5) + n_s^2 =$$

$$(n_d + n_s)\,(n_d + n_s + 5) = N(N + 5). \tag{14.36}$$

Because $\boldsymbol{N}$ is the number operator for bosons, the operator $\boldsymbol{C}_{2,\,u(6)}$ has the eigenvalue $N(N + 5)$ and it meets the conditions (13.31) up to (13.33). According to (14.36) one can write $\boldsymbol{N}^2 = \boldsymbol{C}_{u(6)} - 5\boldsymbol{N}$. The operator $\boldsymbol{N}$ is the linear Casimir operator of $u(6)$.

In table 14.2 the Casimir operators of $\boldsymbol{H}^{(\mathrm{I})}$ (14.31) are put together.

Table 14.2

Operators included in the Hamilton operator of the vibrational limit	Symbols for these Casimir operators	Eigenvalues
$\boldsymbol{J}^2 = -10\sqrt{3}\cdot[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}]^{(0)}$	$\boldsymbol{C}_{2,so\,(3)}$	$J(J + 1)$
$\boldsymbol{T}^2 = - 2 \sum_{J = 1,\,3} \sqrt{(2J + 1)}\cdot[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J)}]^{(0)}$	$\boldsymbol{C}_{2,so\,(5)}$	$\tau(\tau + 3)$
$\boldsymbol{n_d}(\boldsymbol{n_d} + 4) = \sum_{k = 0}^{4} (-1)^k\sqrt{(2k + 1)}[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)}]^{(0)}$	$\boldsymbol{C}_{2,u\,(5)}$	$n_d(n_d + 4)$
$\boldsymbol{n_d} = \sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}$	$\boldsymbol{C}_{1,u\,(5)}$	n_d
$\boldsymbol{N}(\boldsymbol{N} + 5) = \sum_{k=0}^{4} (-1)^k \sqrt{(2k + 1)}\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)}]^{(0)} + \sqrt{5}\, [[\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)} \times [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}]^{(0)} + \sqrt{5}\, [[\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)} \times [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}]^{(0)} + [[\boldsymbol{s}^+, \boldsymbol{s}]^{(0)} \times [\boldsymbol{s}^+, \boldsymbol{s}]^{(0)}]^{(0)}$	$\boldsymbol{C}_{2,u\,(6)}$	$N(N + 5)$
$\boldsymbol{N} = [\boldsymbol{s}^+, \boldsymbol{s}]^{(0)} + \sqrt{5}[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(0)}$	$\boldsymbol{C}_{1,u\,(6)}$	N

The operator $\boldsymbol{H}^{(\mathrm{I})}$ (14.31) can be written like this as well

$$\boldsymbol{H}^{(\mathrm{I})} =$$

$$\alpha_1\, \boldsymbol{C}_{1,u\,(6)} + \alpha_2\, \boldsymbol{C}_{2,u\,(6)} + \beta\, \boldsymbol{C}_{1,u\,(6)}\, \boldsymbol{C}_{1,u\,(5)} + \gamma_1\, \boldsymbol{C}_{1,u\,(5)} + \gamma_2\, \boldsymbol{C}_{2,u\,(5)} + \delta\, \boldsymbol{C}_{2,so\,(5)} + \varepsilon\, \boldsymbol{C}_{2,so\,(3)}$$

with $\alpha_1 = \varepsilon_n - 5v_n,\ \alpha_2 = v_n,\quad \beta = v_{nd},\quad \gamma_1 = \varepsilon'_d - 4v_d,$ (14.37)

$\gamma_2 = v_d,\quad \delta = v_t,\quad \varepsilon = v_j.$

With the help of table 14.2 the eigenenergy can be represented this way

$$E^{(\mathrm{I})} = \tag{14.38}$$

$$\alpha_1 N + \alpha_2 N(N + 5) + \beta\, N\, n_d + \gamma_1 n_d + \gamma_2 n_d(n_d + 4) + \delta\, \tau(\tau + 3) + \varepsilon\, J(J + 1).$$

In section 10.3 eigenvalues of the Hamilton operator $\boldsymbol{H}^{(\mathrm{I})}$ have been calculated with (10.17) and compared with measured level energies.

Whenever it is possible to represent energy eigenvalues in terms of eigenvalues of Casimir operators there is talk of **dynamical symmetry**.

14.5 Casimir operators of the *su*(3)- or rotational limit

According to (14.12) the operators

$$\boldsymbol{J}_\kappa = [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(1)}{}_\kappa \text{ and}$$

$$\boldsymbol{Q}_\mu = [\boldsymbol{d}^+ \times \boldsymbol{s}]^{(2)}{}_\mu + [\boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)}{}_\mu - (\sqrt{7}/2)\,[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)}{}_\mu \qquad (14.39)$$

are basis elements of the *su*(3) algebra in the rotational limit ((14.17) and table 14.1). In order to find its Casimir operator we look into the commutator of $\boldsymbol{Q}^2 = \sum_\lambda (-1)^\lambda \boldsymbol{Q}_\lambda \boldsymbol{Q}_{-\lambda}$ (9.4) and $\boldsymbol{Q}_\mu$ and maintain

$$[\boldsymbol{Q}^2, \boldsymbol{Q}_\mu] = 3\sqrt{(3/32)}\, \sum_{\mu'\mu''} (1\ \mu'\ 2\ \mu'' \mid 2\ \mu)\, (\boldsymbol{J}_{\mu'} \boldsymbol{Q}_{\mu''} + \boldsymbol{Q}_{\mu''} \boldsymbol{J}_{\mu'}). \qquad (14.40)$$

Owing to (14.16) and (14.14) the relations

$$[\boldsymbol{Q}_\kappa, \boldsymbol{Q}_\mu] = 3\sqrt{(5/32)}\, (2\ \kappa\ 2\ \mu \mid 1,\ \kappa + \mu)\, \boldsymbol{J}_{\kappa+\mu}$$

$$[\boldsymbol{Q}_\kappa, \boldsymbol{J}_\mu] = -\sqrt{6}\, (2\ \kappa\ 1\ \mu \mid 2,\ \kappa + \mu)\, \boldsymbol{Q}_{\kappa+\mu} \qquad (14.41)$$

are valid. We insert them in succession in $[\boldsymbol{Q}^2, \boldsymbol{Q}_\mu]$ and obtain the equation (14.40) making use of (A1.4) and (A1.6). Analogously one obtains

$$[\boldsymbol{J}^2, \boldsymbol{Q}_\mu] = -\sqrt{6}\, \sum_{\mu'\mu''} (1\ \mu'\ 2\ \mu'' \mid 2\ \mu)\, (\boldsymbol{J}_{\mu'} \boldsymbol{Q}_{\mu''} + \boldsymbol{Q}_{\mu''} \boldsymbol{J}_{\mu'}). \qquad (14.42)$$

One multiplies this relation by 3/8, adds it to (14.40) and obtains

$$[\boldsymbol{Q}^2 + (3/8)\boldsymbol{J}^2, \boldsymbol{Q}_\mu] = 0. \qquad (14.43)$$

The operator $(1/2)\boldsymbol{C}_{2,\,su(3)} = \boldsymbol{Q}^2 + (3/8)\boldsymbol{J}^2$ (14.44)

commutes not only with $\boldsymbol{Q}_\mu$ but also with $\boldsymbol{J}_\mu$, because on the one hand $\boldsymbol{J}^2$ commutes anyway with $\boldsymbol{J}_\mu$ (13.28) and on the other hand one can show that $\boldsymbol{Q}^2$ commutes with $\boldsymbol{J}_\mu$. That is, with the aid of (14.41), (A1.4) and (A1.6) one obtains

$$\begin{aligned}
[\boldsymbol{Q}^2, \boldsymbol{J}_\mu] = &-\sqrt{6}\, \sum_\lambda (-1)^\lambda (2, -\lambda, 1\ \mu \mid 2, -\lambda + \mu)\, \boldsymbol{Q}_\lambda \boldsymbol{Q}_{-\lambda+\mu} - \\
&\sqrt{6}\, \sum_\lambda (-1)^\lambda (2\ \lambda\ 1\ \mu \mid 2,\ \lambda + \mu)\, \boldsymbol{Q}_{\lambda+\mu} \boldsymbol{Q}_{-\lambda} = \qquad (14.45)\\
&-\sqrt{6}\,\sqrt{(5/3)}\, \Big(\sum_{\lambda'\lambda''} (2, -\lambda', 2\ -\lambda'' \mid 1, -\mu)\, \boldsymbol{Q}_{\lambda'} \boldsymbol{Q}_{\lambda''} + \\
&\sum_{\lambda'\lambda''} (2, -\lambda'', 2\ -\lambda' \mid 1, -\mu)\, \boldsymbol{Q}_{\lambda'} \boldsymbol{Q}_{\lambda''}\Big) = 0.
\end{aligned}$$

Thus the operator (14.44) commutes with all elements (14.39) of the *su*(3) algebra and represents its Casimir operator. Its structure meets the condition (13.32). The factor ½ in (14.44) gives the Casimir operator the usual form. According to (13.35) the operator $\boldsymbol{C}_{2,\,su(3)}$ has the following eigenvalues

$\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)$, λ, μ integers

when it acts on the corresponding eigenstates (provided that they are in a so-called irreducible representation) which are themselves characterised by the parameters λ and μ. Naturally these states are also eigenfunctions of $\boldsymbol{J}^2$ with the eigenvalue $J(J + 1)$. A detailed treatment of these eigenstates reveals that there exist restrictions for λ and μ. They are positive integers and obey the following rules

$\mu = 0, 2, 4, \ldots$

$\lambda = 2N - 6l - 2\mu$ with $l = 0, 1, \ldots$ and N = number of bosons. (14.46)

Therefore, the doublets (λ, μ) contain the following values: for N = 1 they amount to (2, 0); for N = 2 we have (4, 0), (0, 2); for N = 3 consequently (6, 0), (2, 2), (0, 0) and so forth. Further doublets can be taken from figure 14.1.

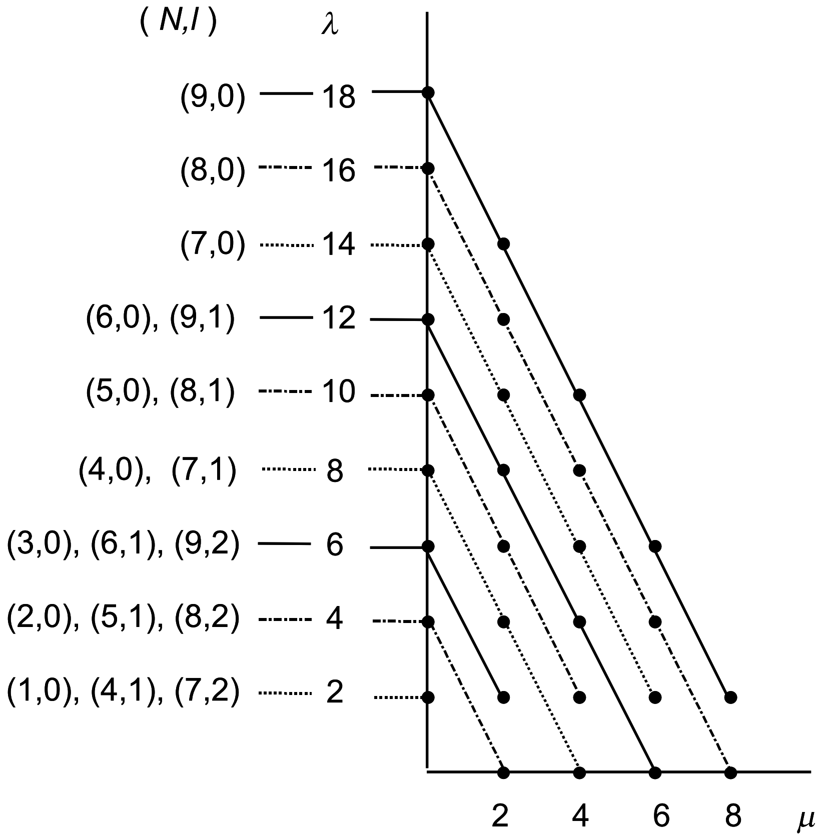

Figure 14.1. Graph of the doublets (λ,μ) according to the equations (14.46). They represent descending straight lines. At their highest point the corresponding doublets (N,l) are given.

A given doublet (λ, μ) admits only selected values for J. They depend on a ordering number K, for which the following values are permitted

$$K = 0, 2, 4, \ldots, \min(\lambda, \mu). \qquad (14.47)$$

For $K = 0$ the values $J = 0, 2, 4, \ldots, \max(\lambda, \mu)$ are allowed

and for $K > 0$ $\qquad J = K, K + 1, K + 2, \ldots, K + \max(\lambda, \mu)$. (14.48)

We look at the simplest values. For $\mu = l = 0$ we have $(\lambda, \mu) = (2N, 0)$ and the series of the ground state shows the values $J = 0, 2, 4, .., 2N$. The case $\mu = 2$, $l = 0$ is the next higher irreducible representation with $(\lambda, \mu) = (2N - 4, 2)$ where the values $K = 0$ with $J = 0, 2, 4, .., 2N - 2$ and $K = 2$ with $J = 2, 3, 4, .., 2N - 2$ arise. It happens that J-values which belong to the same doublet (λ, μ) appear many times. We will show that such states have the same energy i. e. they are degenerated. The parameter K is needed for describing the eigenstates.

The *su*(3)- or rotational limit is realised by deleting the following coefficients in the Hamiltonian (9.39)

$$\varepsilon_d' = v_{nd} = v_d = v_r = v_t = 0. \qquad (14.49)$$

The remaining Hamiltonian consists of the Casimir operators of the second chain of algebras ((14.17) and table 14.1) as follows

$$\boldsymbol{H}^{(\mathrm{II})} = \varepsilon_n \boldsymbol{N} + v_n \boldsymbol{N}^2 + v_j \boldsymbol{J}^2 + \boldsymbol{v_q Q}^2. \qquad (14.50)$$

We write the last two terms like this

$$v_j \boldsymbol{J}^2 + \boldsymbol{v_q Q}^2 = \qquad (14.51)$$

$$v_j \boldsymbol{J}^2 - (3/8)v_q \boldsymbol{J}^2 + v_q \boldsymbol{Q}^2 + (3/8)v_q \boldsymbol{J}^2 = (v_j - (3/8)v_q)\boldsymbol{J}^2 + (1/2)v_q \boldsymbol{C}_{su(3)}.$$

The eigenstates of the Casimir operator of the *su*(3) chain are eigenstates of the Hamilton operator $\boldsymbol{H}^{(\mathrm{II})}$ as well. The operators $\boldsymbol{N}$ and $\boldsymbol{N}(\boldsymbol{N} + 5) = \boldsymbol{C}_{u(6)}$ have already been discussed (14.36). Thus the eigenvalues of $\boldsymbol{H}^{(\mathrm{II})}$ read

$$E^{(\mathrm{II})} = \qquad (14.52)$$

$$\varepsilon_n N + v_n N^2 + (v_j - (3/8)v_q)J(J + 1) + (1/2)\, v_q (\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)).$$

They don't depend directly on the ordering number K (14.47). Figure 14.2 shows a typical energy spectrum of a nucleus of the *su*(3) limit, which is calculated with the help of (14.52). Because of $v_q < 0$ the level energies of every series with given (λ, μ) ascend according to the term $(v_j - (3/8)v_q)J(J + 1)$. This behaviour is typical for the rotator and that's why the *su*(3) limit is named rotational limit as well.

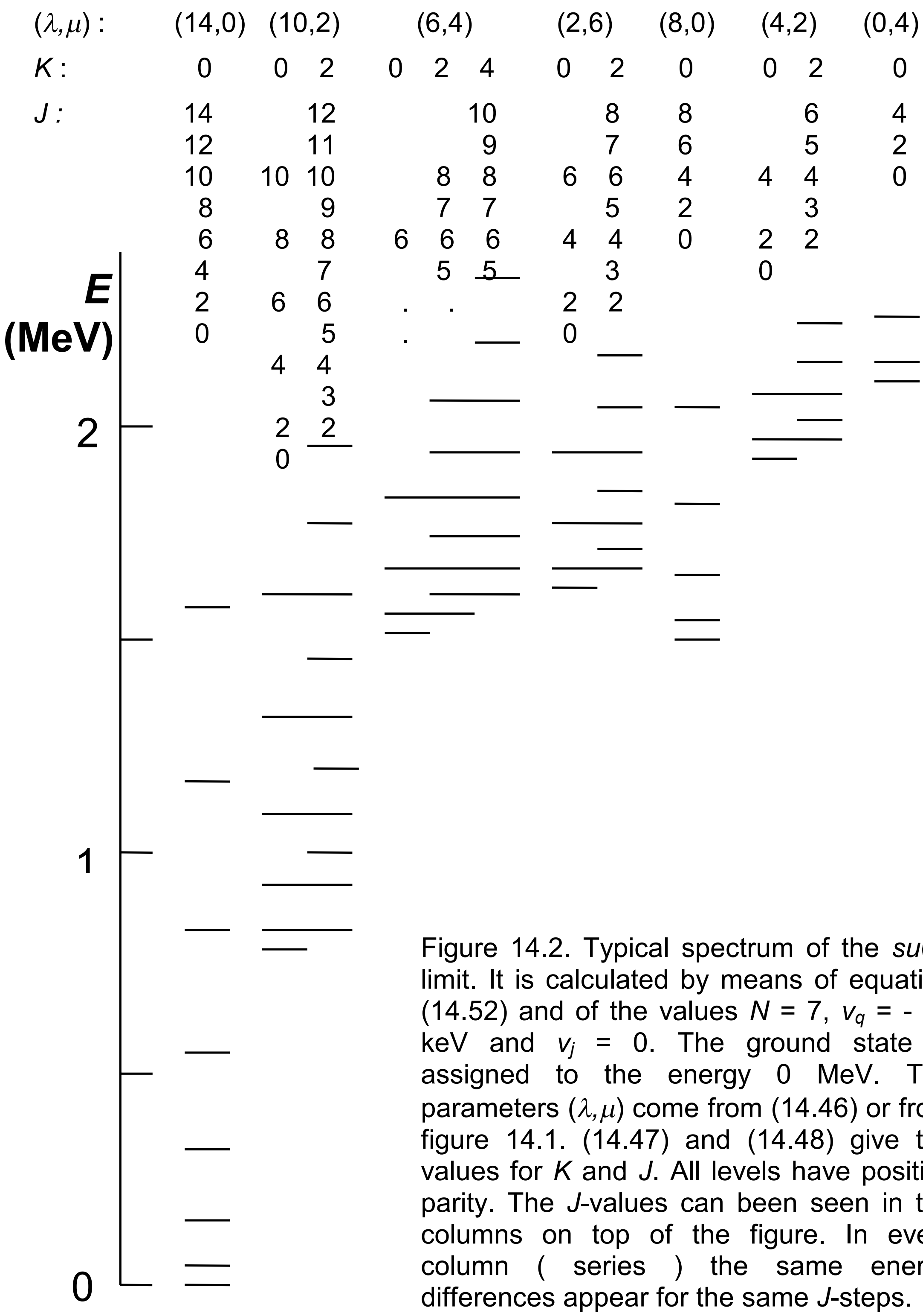

Figure 14.2. Typical spectrum of the *su*(3) limit. It is calculated by means of equation (14.52) and of the values N = 7, v_q = - 20 keV and v_j = 0. The ground state is assigned to the energy 0 MeV. The parameters (λ,μ) come from (14.46) or from figure 14.1. (14.47) and (14.48) give the values for K and J. All levels have positive parity. The J-values can been seen in the columns on top of the figure. In every column (series) the same energy differences appear for the same J-steps.

Extensive investigations (Arima and Iachello, 1978, 1987) reveal that the nuclei which behave like the *su*(3) model have proton- and neutron numbers deviating clearly from the magic numbers 82 and 126 (figure 14.3).

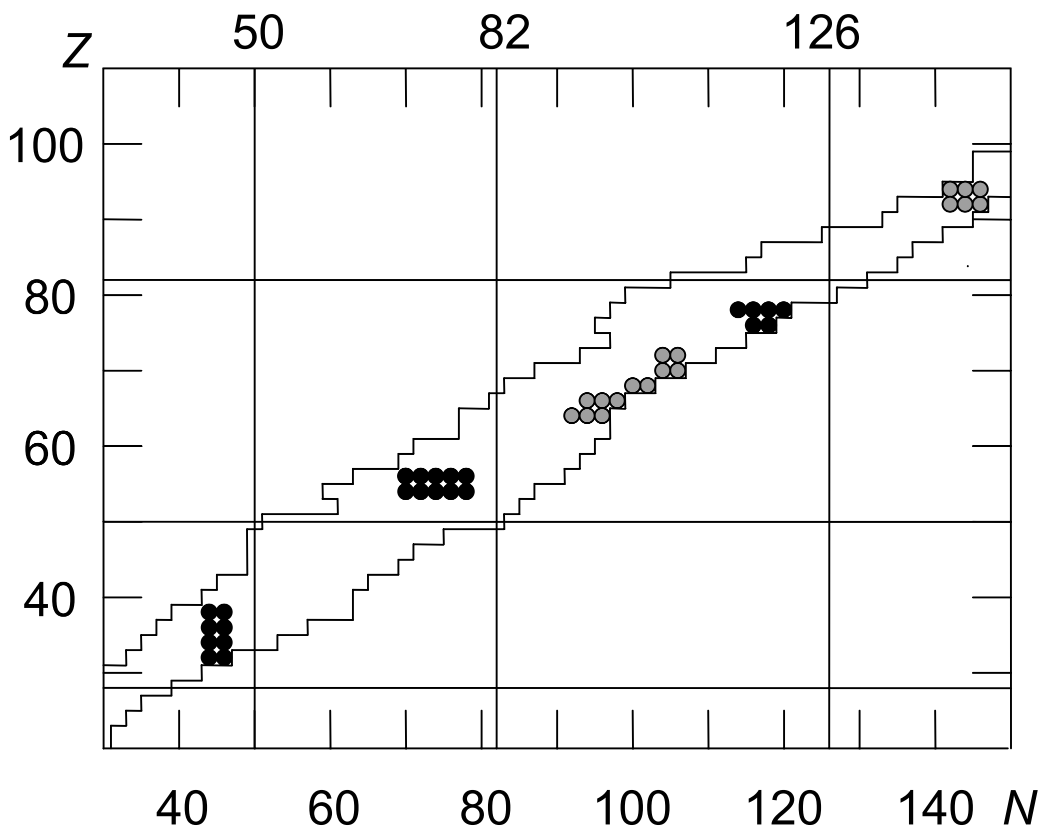

Figure 14.3. Card of even-even nuclei. *Z* = number of protons, *N* = number of neutrons. The grey circles denote nuclei of the *su*(3) limit and the black ones are representations of the *so*(6) limit (Iachello and Arima, 1987, p.42).

Figure 14.4 shows that the rare earth ^{170}Er corresponds well to the energy scheme of the *su*(3) limit.

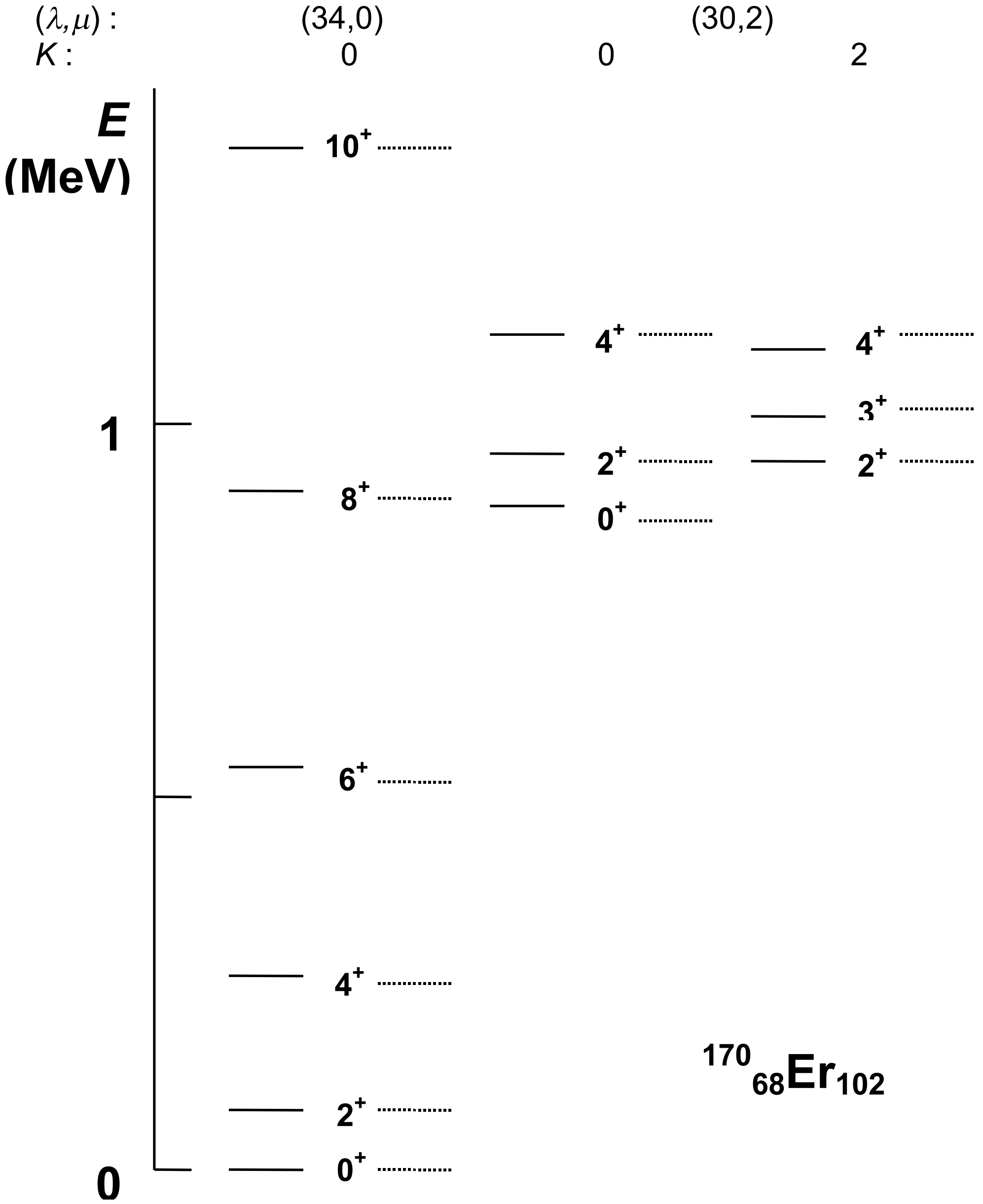

Figure 14.4. Comparison between the measured and the calculated *su*(3)-spectrum of ^{170}Er (Arima and Iachello, 1978, p. 215). The parameters N = 17, v_j = 9.19 keV and v_q = - 8.8 keV have been used with formula (14.52). Broken lines represent the calculated energies.

14.6 Casimir operators of the *so*(6)- or γ-instable limit

The chain of Lie algebras (14.21) defining the *so*(6) limit comprises the subalgebras *so*(5) $\supset$ *so*(3). Their Casimir operators have been dealt with in the elements (14.18), has the following Casimir operator (Talmi, 1993, p.778)

$$\boldsymbol{C}_{\boldsymbol{so(6)}} = 2 \sum_{k=1,3} \sum_{\kappa} (-1)^{\kappa} [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(k)}{}_{\kappa} [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(k)}{}_{-\kappa} + \sum_{\lambda} (-1)^{\lambda} \boldsymbol{S}_{\lambda} \boldsymbol{S}_{-\lambda} \text{ with}$$

$$\boldsymbol{S}_{\lambda} = [\boldsymbol{d}^{+} \times \boldsymbol{s}]^{(2)}{}_{\lambda} + [\boldsymbol{s}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}{}_{\lambda}. \qquad (14.53)$$

Its form meets the condition (13.32). We have to check if $\boldsymbol{C}_{\boldsymbol{so(6)}}$, which is identical with $\mathbf{R}^2$ (14.7), commutes with the operators (14.18). Because the first part of $\boldsymbol{C}_{\boldsymbol{so(6)}}$ is identical with $\boldsymbol{T}^2$ according to (14.5), this part commutes with the affiliated elements $[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(k)}{}_{\kappa}$ (k = 1, 3) (see 14.32). But these operators commute also with the second part of $\boldsymbol{C}_{\boldsymbol{so(6)}}$, i. e. with $\sum_{\lambda} (-1)^{\lambda} \boldsymbol{S}_{\lambda} \boldsymbol{S}_{-\lambda}$, which can be shown with the aid of (14.20) and roughly with the procedure employed for (14.45). With the same methods one proves that $\boldsymbol{S}_{\lambda}$ commutes with $\boldsymbol{C}_{\boldsymbol{so(6)}}$ and in doing so, one finds that the following expressions cancel out

$$[\boldsymbol{T}^2, \boldsymbol{S}_{\kappa}] + \sum_{\lambda} (-1)^{\lambda} [\boldsymbol{S}_{\lambda} \boldsymbol{S}_{-\lambda}, \boldsymbol{S}_{\kappa}] = 0, \text{ i. e. } [\boldsymbol{C}_{\boldsymbol{so(6)}}, \boldsymbol{S}_{\kappa}] = 0. \qquad (14.54)$$

Consequently the operator $\boldsymbol{C}_{\boldsymbol{so(6)}}$ meets the condition (13.31). It is written in the standard form and therefore its eigenvalue reads according to (13.34) (rank N = 6)

$$\rho\,(\rho + 4). \qquad (14.55)$$

A detailed investigation of the eigenstates of the *so*(6) algebra reveals that the admissible values for ρ are

$$\rho = N, N - 2, N - 4, ..., 1 \text{ or } 0 \qquad (N : \text{number of bosons}). \qquad (14.56)$$

Moreover, for the parameter τ of the Casimir operator $\boldsymbol{T}^2$ the following values are admitted

$$\tau = 0, 1, ... , \rho. \qquad (14.57)$$

According to (10.5) J can amount to

$$J = \lambda, \lambda + 1, ... , 2\lambda - 3, 2\lambda - 2, 2\lambda \qquad (14.58)$$

with $\lambda = \tau - 3n_{\Delta}$ and n_{Δ} = 0, 1, ... (see chapter 3). Table 14.2 gives permissible values for J.

Table 14.2. Values for J of the $so(6)$ limit depending on n_Δ and τ.

τ	n_Δ	J
0	0	0
1	0	2
2	0	2, 4
3	1	0
	0	3, 4, 6
4	1	2
	0	4, 5, 6, 8
5	1	2, 4
	0	5, 6, 7, 8, 10
6	2	0
	1	3, 4, 6
	0	6, 7, 8, 9, 10, 12

One treats the Casimir operator of the highest algebra, $u(6)$, in the same way as in the previously discussed special cases of the IBM1.

Now we turn to the Hamilton operator. For the present special case, the $so(6)$ limit, one deletes the following coefficients of the equation (9.39)

$$\varepsilon_d' = v_{nd} = v_d = v_q = 0. \tag{14.59}$$

Choosing $v_q = 0$ results in $v_2 = 0$ (9.38), which means (see 7.17) that the Hamilton operator is not able any longer to interchange *dd*- with *sd*-states i. e. that there is no interaction between these states. The remaining Hamilton operator, which defines the $so(6)$- or γ-instable limit, reads

$$\boldsymbol{H}^{(\mathrm{III})} = \varepsilon_n \boldsymbol{N} + v_n \boldsymbol{N}^2 + v_r \boldsymbol{R}^2 + v_t \boldsymbol{T}^2 + v_j \boldsymbol{J}^2. \tag{14.60}$$

It represents a linear combination of the Casimir operators of the chain (14.21). Again the functions of the highest algebra are functions of the lower ones and therefore the energy eigenvalue is given according to (10.3), (10.13) and (14.55) by the following expression

$$E^{(\mathrm{III})} = \varepsilon_n N + v_n N^2 + v_r \rho\,(\rho + 4) + v_t\, \tau\,(\tau + 3) + v_j J(J + 1). \tag{14.61}$$

Figure 14.5 shows a typical energy spectrum of a nucleus of the $so(6)$ limit according to equation (14.61). The 2^+- 4^+-doublet of the third and fourth level (with $\tau = 2$) is characteristic for this special case and appears also in the so-called γ-instable nuclei, for which reason this model is named γ-instable limit as well.

In figure 14.3 the regions with nuclei of the $so(6)$ type are marked with black dots. They join always regions with $su(5)$ nuclei (see figure 10.7) which corresponds to the noticeable similarity of both Hamiltonians.

(ρ,n_Δ) :	(6,0)	(6,1)	(6,2)	(4,0)	(4,1)	(2,0)	(0,0)
	τ; --------J----------	τ; ---J----	τ; J	τ; ----J------	τ; J	τ; -J--	τ; J
	6;12,10, 9, 8, 7, 6	6; 6, 4, 3	6; 0	4; 8, 6, 5, 4	4; 2	2; 4, 2	0; 0
	5;10, 8, 7, 6, 5	5; 4, 2		3; 6, 4, 3	3; 0	1; 2	
	4; 8, 6, 5, 4	4; 2		2; 4, 2		0: 0	
	3; 6, 4, 3	3; 0		1; 2			
	2; 4, 2			0; 0			
	1; 2						
	0; 0						

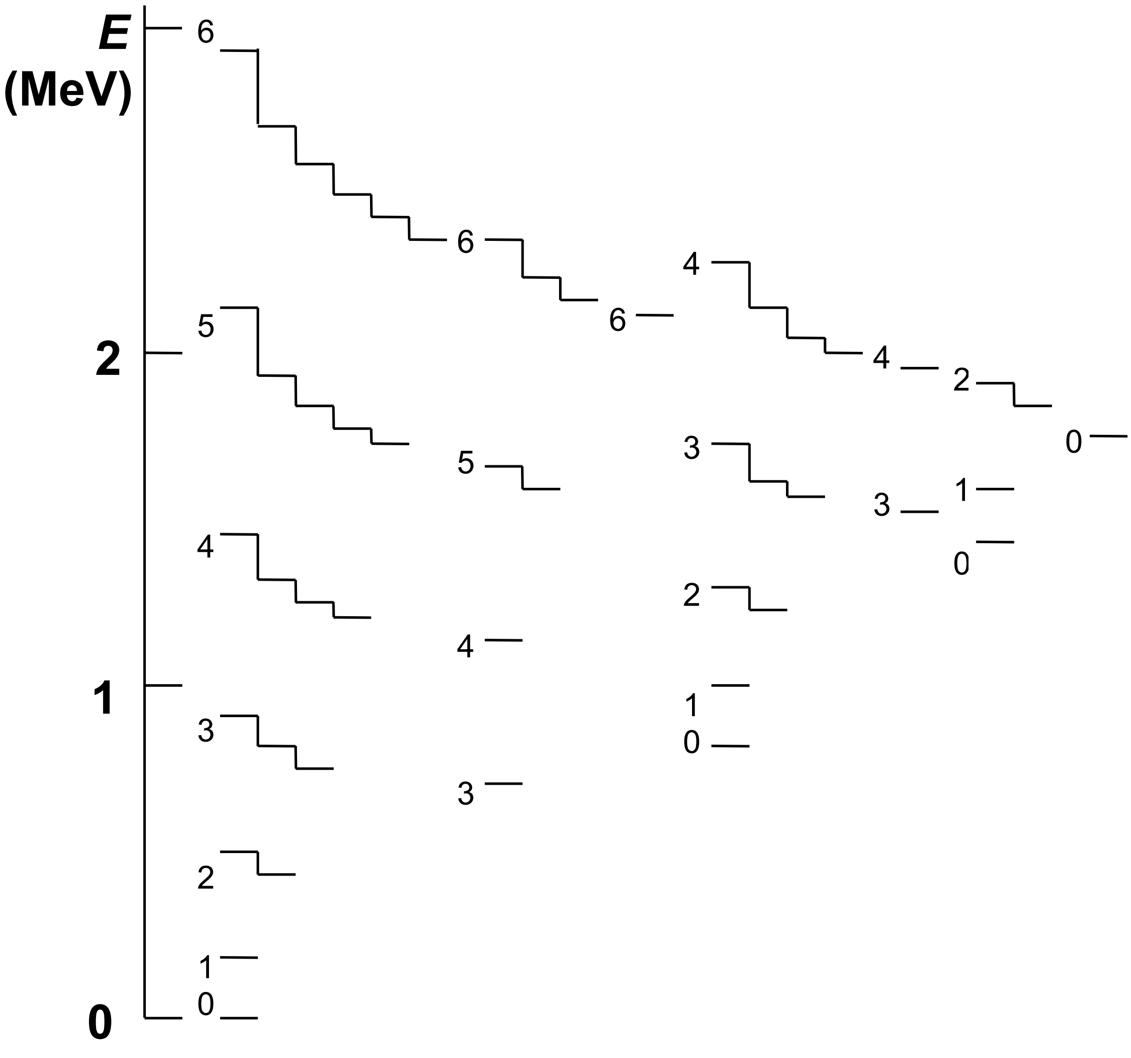

Figure 14.5. Typical spectrum of the so(6) limit. It is calculated by means of equation (14.61) and the values N = 6, v_r = - 30 keV, v_t = 40 keV and v_j = 5,1 keV. The parameters ρ and τ follow the rules (14.56) and (14.57). The values of n_Δ and J are given by (14.58). All levels have positive parity. Level groups with the same τ are joined together and labelled with the value of τ. The J-values can be taken from the columns on top of the figure.

Figures 14.6 and 14.7 compare measured with calculated levels of ^{196}Pt and ^{132}Ba.

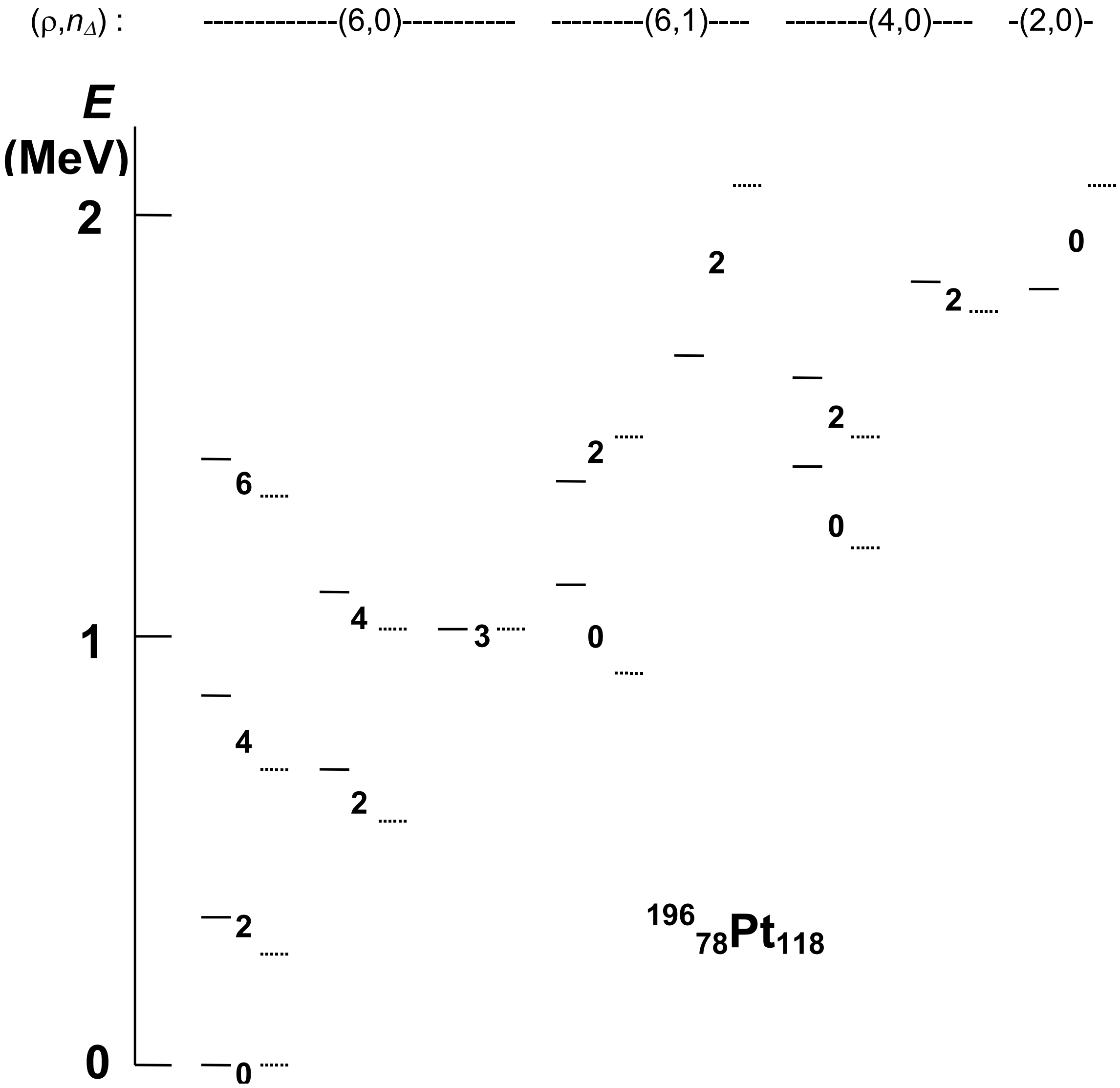

Figure 14.6. Comparison between the measured and the calculated *so*(6)-spectrum of ^{196}Pt (Iachello and Arima, 1987, p. 43). The parameters $N = 6$, $v_r = -42.75$ keV, $v_t = 50$ keV and $v_j = 10$ keV have been used with formula (14.61). The spins are given and the parity is positive. Broken lines represent calculated energies.

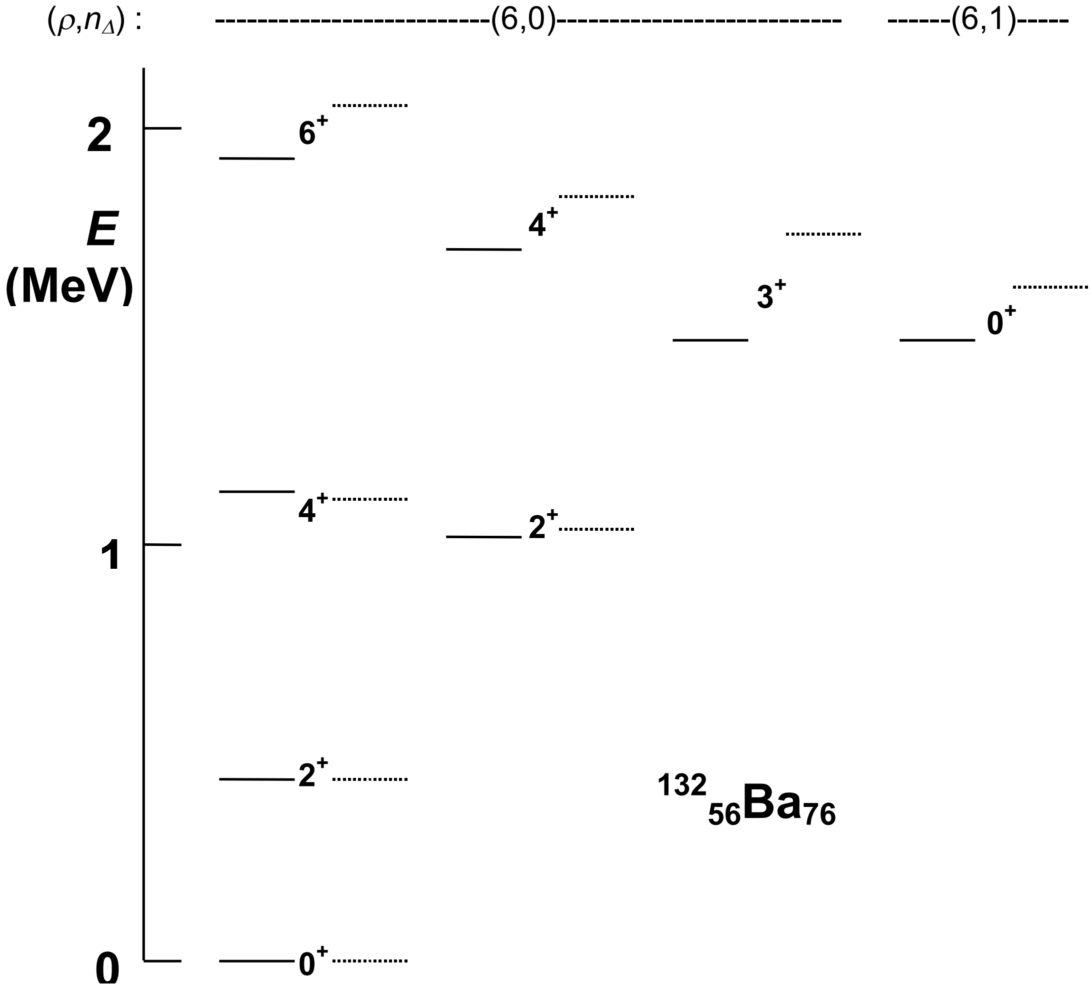

Figure 14.7. Comparison between the measured and the calculated *so*(6)-spectrum of ^{132}Ba (Arima and Iachello, 1979, p. 486). The expression (14.61) and the parameters v_t = 90 keV and v_j = 10 keV have been used. Because only the states with $\rho = N = 6$ have been identified unambiguously, v_r does not appear. Broken lines represent the calculated energies.

The *so*(6) limit offers the chance to test the program PHINT (appendix 7). One chooses first the parameters in equation (14.60) and puts $\varepsilon_n = v_n = 0$, which is admissible when the spectrum of a single nucleus is looked at. By inserting in equation (9.38) and by transforming one obtains

$$
\begin{aligned}
\varepsilon_d - \varepsilon_s &= 2\,(\,3v_j + 2v_t\,)\,, \\
c_0 &= 4\,(\,-\,3v_j - 2v_t - 2v_r\,), \\
c_2 &= 2\,(\,-3v_j + v_t + v_r\,), \\
c_4 &= 2\,(\,4v_j + v_t + v_r\,), \\
v_2 &= 0, \\
v_0 &= 2\,\sqrt{5}v_r\,, \\
u_2 &= 2\,\sqrt{5}v_r\,, \\
u_0 &= 0.
\end{aligned}
\tag{14.62}
$$

By means of (A7.1) one finds the input parameters for the program. Calculations using the parameters of figure 14.5, which are performed both according to (14.61) and with the help of PHINT and (14.62), agree perfectly.

According to Iachello and Arima (1987, p. 25) it can be shown that there exist only three chains of algebras, (14.11), (14.17) and (14.21), in the interacting boson model 1. Therefore there are only three types of dynamical symmetries.

15 The proton-neutron interacting boson model IBM2

In this interacting boson model, protons and neutrons are treated separately. In the sections 1 up to 3, we will show how the Hamilton operator of this model can be simplified and how the number of parameters is reduced. Sections 4 and 5 demonstrate the eigenenergies and transition probabilities and compare them with experimental data. There will be talk of group theoretical structures of the IBM2 in section 6.

15.1 The complete Hamiltonian of the IBM2

In the IBM1 the boson number of the protons is added to the boson number of neutrons (chapter 2) and later on, no distinction is made between these types of bosons. In chapter 7 the interaction terms don't contain Coulomb forces. Moreover, speaking in terms of the shell model, the fact is known that in intermediate and heavy nuclei protons and neutrons occupy different shells. This results in an interaction between proton- and neutron pairs, which differs from the interaction between identical pairs. This fact contrasts to the well-known charge independence of the short-range nuclear forces. Likewise, this effect is not taken into account in the IBM1.

Because of these simplifications, the parameters of the IBM1 fitted on experimental data change from nucleus to nucleus in a discontinuous and hardly interpretable way. On the other hand the interacting model 2, which distinguishes between proton- (π) and neutron (ν) bosons, yields results which can be interpreted by means of the SD-version of the shell model. The IBM2 has special practical significance because its parameters depend in a smoothed curve on π- and ν-boson numbers and it makes possible to calculate unknown nuclear spectra (Arima and Iachello, 1984, p. 177). The interest in the algebraic structure of the IBM2 has even grown since states with special magnetic dipole properties (Bohle, 1984) are known which can be explained with group theoretical methods.

The numbers of π-bosons N_π and ν-bosons N_ν are fixed equally as in chapter 2. There is no boson composed of a proton and a neutron. There exist 12 creation operators for bosons

$$\boldsymbol{b}^{+}_{\pi,\,jm} = \boldsymbol{s}^{+}_{\pi},\ \boldsymbol{d}^{+}_{\pi,\,m}\ (m = -2, -1, ..,2) \text{ and}$$

$$\boldsymbol{b}^{+}_{\nu,\,jm} = \boldsymbol{s}^{+}_{\nu},\ \boldsymbol{d}^{+}_{\nu,\,m}\ (m = -2, -1, ..,2). \qquad (15.1)$$

Analogously to (6.19), we have the same number of tensor operators for boson annihilation like this

$$\boldsymbol{b}^{\sim}_{\pi,\,jm} = \boldsymbol{s}_{\pi},\ \boldsymbol{d}^{\sim}_{\pi,\,m},\ \ \boldsymbol{b}^{\sim}_{\nu,\,jm} = \boldsymbol{s}_{\nu},\ \boldsymbol{d}^{\sim}_{\nu,\,m}\ (m = -2, -1, ..,2). \qquad (15.2)$$

In analogy with (5.33) and (5.35), every *d*- or *s*-operator for protons commutes with every *d*- or *s*-operator for neutrons. We demand that both N_π and N_ν are good quantum numbers i. e. the Hamilton operator must meet the condition

$$[\boldsymbol{H}\,,\,\boldsymbol{N}_\pi] = [\boldsymbol{H}\,,\,\boldsymbol{N}_\nu] = 0 \tag{15.3}$$

with the operators $\boldsymbol{N}_\pi = \sqrt{5}[\boldsymbol{d}^{+}{}_{\pi} \times \boldsymbol{d}^{\sim}{}_{\pi}]^{(0)} + \boldsymbol{s}^{+}{}_{\pi}\,\boldsymbol{s}_\pi$ (6.37) and the corresponding $\boldsymbol{N}_\nu$. Generalising the equations (7.3), (7.5) and (7.6), we write the Hamilton operator as follows

$$\boldsymbol{H} = \boldsymbol{H}_\pi + \boldsymbol{H}_\nu + \boldsymbol{V}_{\pi\nu}\,. \tag{15.4}$$

Both $\boldsymbol{H}_\pi$ and $\boldsymbol{H}_\nu$ have the form of the IBM1-Hamiltonian (7.3, 7.6, and 7.17) but there is nothing than π-operators in $\boldsymbol{H}_\pi$ and ν-operators in $\boldsymbol{H}_\nu$. We write the mixing potential operator $\boldsymbol{V}_{\pi\nu}$ employing the parameter

$$\rho = \pi,\ \nu \tag{15.5}$$

analogously to (7.6) like this

$$\boldsymbol{V}_{\pi\nu} = (1/2) \sum_{j_f\, j_g\, j_p\, j_q\, m_f\, m_g\, m_p\, m_q} \sum_{\rho_f \neq \rho_g} \sum_{\rho_p \neq \rho_q} \tag{15.6}$$

$$\langle \rho_f\, j_f\, m_f\,,\ \rho_g\, j_g\, m_g \mid \boldsymbol{W} \mid \rho_p\, j_p\, m_p\,,\ \rho_q\, j_q\, m_q \rangle \cdot \boldsymbol{b}^{+}{}_{\rho_f\, j_f\, m_f}\, \boldsymbol{b}^{+}{}_{\rho_g\, j_g\, m_g}\, \boldsymbol{b}_{\rho_p\, j_p\, m_p}\, \boldsymbol{b}_{\rho_q\, j_q\, m_q}$$

where $j = 0$ denotes a *s*-operator and $j = 2$ a *d*-operator. Both the creation and the annihilation operators in (15.6) comprise a π- and a ν-operator ($\rho_f \neq \rho_g$, $\rho_p \neq \rho_q$), otherwise the matrix element vanishes. Analogously to (7.6) $\boldsymbol{V}_{\pi\nu}$ can be written by means of vector coupling as

$$\boldsymbol{V}_{\pi\nu} = (1/2) \sum_{j_f\, j_g\, j_p\, j_q} \sum_{J=0}^{4} \sum_{\rho_f \neq \rho_g} \sum_{\rho_p \neq \rho_q} \langle \rho_f\, j_f,\ \rho_g\, j_g,\ J \mid \boldsymbol{W} \mid \rho_p\, j_p,\ \rho_q\, j_q,\ J \rangle \cdot$$

$$\sqrt{(2J + 1)}\ [[\boldsymbol{b}^{+}{}_{\rho_f\, j_f} \times \boldsymbol{b}^{+}{}_{\rho_g\, j_g}]^{(J)} \times [\boldsymbol{b}^{\sim}{}_{\rho_p\, j_p} \times \boldsymbol{b}^{\sim}{}_{\rho_q\, j_q}]^{(J)}]^{(0)}. \tag{15.7}$$

Because the operators $\boldsymbol{b}^{+}{}_{\pi\, j\, m}$ and $\boldsymbol{b}^{+}{}_{\nu\, j\, m}$ create distinguishable bosons, the common state does not show symmetry according to (A2.5) and for $[\boldsymbol{d}^{+}{}_{\pi} \times \boldsymbol{d}^{+}{}_{\nu}]^{(J)}$ the statement (6.6) is not valid any more. On the contrary, for J all integers from 0 up to 4 are possible.

In the case of $j_p = j_q = 2$ for the two-boson state in the matrix element in (15.7) one obtains the following expression

$$|\ \pi\, 2,\ \nu\, 2,\ J\, M \rangle = \sum_{m_\pi\ m_\nu} (2\ m_\pi\ 2\ m_\nu \mid J\, M\,)\ |\ \pi\, 2\ m_\pi,\ \nu\, 2\ m_\nu \rangle = \tag{15.8}$$

$$(-1)^J \sum_{m_\pi\ m_\nu} (2\ m_\nu\ 2\ m_\pi \mid J\, M\,)\ |\ \nu\, 2\ m_\nu\,,\ \pi\, 2\ m_\pi \rangle = (-1)^J\,|\ \nu\, 2,\ \pi\, 2,\ J\, M \rangle.$$

Therefore, there are 12 independent matrix elements in (15.7) as follows

$$W_5 = \langle\, \pi\, 0,\ \nu\, 0,\, 0 \,|\, \boldsymbol{W} \,|\, \pi\, 0,\ \nu\, 0,\, 0 \,\rangle$$
$$W_6 = \langle\, \pi\, 0,\ \nu\, 2,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 0,\ \nu\, 2,\, 2 \,\rangle$$
$$W_7 = \langle\, \pi\, 0,\ \nu\, 2,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 0,\, 2 \,\rangle = \langle\, \pi\, 2,\ \nu\, 0,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 0,\ \nu\, 2,\, 2 \,\rangle$$
$$W_8 = \langle\, \pi\, 2,\ \nu\, 0,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 0,\, 2 \,\rangle \qquad (15.9)$$
$$W_9 = \langle\, \pi\, 0,\ \nu\, 0,\, 0 \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 2,\, 0 \,\rangle = \langle\, \pi\, 2,\ \nu\, 2,\, 0 \,|\, \boldsymbol{W} \,|\, \pi\, 0,\ \nu\, 0,\, 0 \,\rangle$$
$$W_{10} = \langle\, \pi\, 0,\ \nu\, 2,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 2,\, 2 \,\rangle = \langle\, \pi\, 2,\ \nu\, 2,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 0,\ \nu\, 2,\, 2 \,\rangle$$
$$W_{11} = \langle\, \pi\, 2,\ \nu\, 0,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 2,\, 2 \,\rangle = \langle\, \pi\, 2,\ \nu\, 2,\, 2 \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 0,\, 2 \,\rangle.$$

These matrix elements don't alter if one interchanges the π- and the ν-state on one side. The residual elements read

$$W_J = \langle\, \pi\, 2,\ \nu\, 2,\, J \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 2,\, J \,\rangle \quad (\, J = 0, 1, 2, 3, 4\,). \qquad (15.10)$$

Here interchanging the π- and the ν-state results in a factor $(-1)^J$ (see15.8) i. e.

$$\langle\, \pi\, 2,\ \nu\, 2,\, J \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 2,\, J \,\rangle = \langle\, \pi\, 2,\ \nu\, 2,\, J \,|\, \boldsymbol{W} \,|\, \nu\, 2,\ \pi\, 2,\, J \,\rangle\, (-1)^J = \qquad (15.11)$$
$$\langle\, \nu\, 2,\ \pi\, 2,\, J \,|\, \boldsymbol{W} \,|\, \pi\, 2,\ \nu\, 2,\, J \,\rangle\, (-1)^J = \langle\, \nu\, 2,\ \pi\, 2,\, J \,|\, \boldsymbol{W} \,|\, \nu\, 2,\ \pi\, 2,\, J \,\rangle.$$

The partial Hamilton operators $\boldsymbol{H_\pi}$ and $\boldsymbol{H_\nu}$ contain, according to (7.17) and (7.33) respectively, 9 or 6 free parameters each. The considerations in chapter 7 that facilitated the reduction of this number can be applied to the 12 parameters in (15.9) and (15.11) and they can be reduced to 9 (Iachello and Arima, 1987, p. 138). Therefore the Hamilton operator (15.4) comprises 30 or 21 parameters respectively, which render this theory cumbersome and it is practically impossible to fit the parameters unambiguously on experimental data.

15.2 Basis states and the angular momentum operator of the IBM2

Before we deal with the simplification of the Hamilton operator, we turn to the eigenstates. For practical reasons, the basis states of the IBM2 are chosen as vectorial couplings of symmetrical, spherical IBM1-states (6.18) of π- and ν-bosons this way

$$|\, [(n_{s\pi},\, n_{d\pi},\, \tau_\pi,\, n_{\Delta\pi},\, J_\pi) \times (n_{s\nu},\, n_{d\nu},\, \tau_\nu,\, n_{\Delta\nu},\, J_\nu)]^{(J)}{}_M \,\rangle. \qquad (15.12)$$

For example the complete set of states with $N_\pi = N_\nu = 1$ reads as follows

$$|\, [s_\pi \times s_\nu]^{(0)}{}_0 \,\rangle,\ |\, [s_\pi \times d_\nu]^{(2)}{}_\mu \,\rangle,\ |\, [d_\pi \times s_\nu]^{(2)}{}_{\mu'} \,\rangle,\ |\, [d_\pi \times d_\nu]^{(J)}{}_{\mu''} \,\rangle \text{ with}$$
$$(J = 0, 1,..,4). \qquad (15.13)$$

The basis function of the IBM2 (15.12) demands a corresponding angular momentum operator according to (A1.21) and (A1.23). Analogously to (8.9) its components read

$$J_\mu = J_{\pi\mu} + J_{\nu\mu} = \sqrt{10}\,([d^+_\pi \times \tilde{d}_\pi]^{(1)}_\mu + [d^+_\nu \times \tilde{d}_\nu]^{(1)}_\mu) \qquad (15.14)$$

and meet the commutation rules (8.7). According to (A1.22) and (A1.26) the states (15.12) are eigenfunctions of the operators J_0 and $J^2 = J_0^2 - J_0 - 2\,J_1\,J_1$ which are made up by the operators (15.14).

15.3 A reduced Hamiltonian of the IBM2

In section 15.1 we have seen that the IBM2-Hamiltonian comprises a large number of parameters. In practice it is reduced drastically in the course of which one is based on arguments of microscopic theories - mainly on the SD-version of the shell model - and on experiences made by fitting on measured data. A simplification of the Hamilton operator is also necessary because the sum H_π and H_ν generates a large number of energy levels. Because the IBM1 reproduces the spectrum of measured energies relatively well, the IBM2 may not generate essentially more levels.

A first simplification of the Hamiltonian consists in reducing H_π and H_ν to a special case (chapter 14). For that purpose, the vibrational limit (chapter 10) imposes itself because it is able to interpret most nuclei (compare figures 10.7 and 14.3). Thus in both IBM-Hamiltonians on drops R^2 and Q^2 (9.39). In practice in equation (7.17) not only v_0 and v_2 are deleted but also u_0, u_2 and ε_s. Consequently the following expression is left over

$$H_\rho = \varepsilon_\rho\, n_{d\,\rho} + \tfrac{1}{2} \textstyle\sum_{J = 0,\,2,\,4} c_{J\,\rho}\sqrt{(2J + 1)}\, [[d^+_\rho \times d^+_\rho]^{(J)} \times [\tilde{d}_\rho \times \tilde{d}_\rho]^{(J)}]^{(0)}$$

with $\rho = \pi,\ \nu$. (15.15)

The third term $V_{\pi\nu}$ in the Hamilton operator (15.4) is simplified as well. Generally, one employs the following form (Iachello and Arima, 1987, p. 140)

$$V_{\pi\nu} = \kappa\,\sqrt{5}\,[Q_\pi(\chi_\pi) \times Q_\nu(\chi_\nu)]^{(0)} + \lambda\, M_{\pi\nu}. \qquad (15.16)$$

The quadrupole operators $Q_\rho(\chi_\rho)_\mu$ ($\rho = \pi,\ \nu$) correspond to the operators Q_μ in (12.19) that is

$$Q_\rho(\chi_\rho)_\mu = [d^+_\rho \times s_\rho]^{(2)}_\mu + [s^+_\rho \times \tilde{d}_\rho]^{(2)}_\mu + \chi_\rho\,[d^+_\rho \times \tilde{d}_\rho]^{(2)}_\mu\,. \qquad (15.17)$$

For the quadrupole-quadrupole part of $V_{\pi\nu}$ sometimes the symbol

$$\sqrt{5}\,[Q_\pi(\chi_\pi) \times Q_\nu(\chi_\nu)]^{(0)} \equiv Q_\pi(\chi_\pi) \bullet Q_\nu(\chi_\nu). \qquad (15.18)$$

is employed. We write it explicitly

$$\sqrt{5}\,[\boldsymbol{Q}_\pi(\chi_\pi) \times \boldsymbol{Q}_\nu(\chi_\nu)]^{(0)} =$$

$$\begin{aligned}
&\sqrt{5}\,\{([[\boldsymbol{d}^+_\pi \times \boldsymbol{s}_\pi]^{(2)} \times [\boldsymbol{d}^+_\nu \times \boldsymbol{s}_\nu]^{(2)}]^{(0)} + [[\boldsymbol{s}^+_\pi \times \tilde{\boldsymbol{d}}_\pi]^{(2)} \times [\boldsymbol{s}^+_\nu \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)}) + \\
&([[\boldsymbol{d}^+_\pi \times \boldsymbol{s}_\pi]^{(2)} \times [\boldsymbol{s}^+_\nu \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)} + [[\boldsymbol{s}^+_\pi \times \tilde{\boldsymbol{d}}_\pi]^{(2)} \times [\boldsymbol{d}^+_\nu \times \boldsymbol{s}_\nu]^{(2)}]^{(0)}) + \\
&\chi_\nu\,[([\boldsymbol{d}^+_\pi \times \boldsymbol{s}_\pi]^{(2)} + [\boldsymbol{s}^+_\pi \times \tilde{\boldsymbol{d}}_\pi]^{(2)}) \times [\boldsymbol{d}^+_\nu \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)} + \qquad (15.19)\\
&\chi_\pi\,[[\boldsymbol{d}^+_\pi \times \tilde{\boldsymbol{d}}_\pi]^{(2)} \times ([\boldsymbol{d}^+_\nu \times \boldsymbol{s}_\nu]^{(2)} + [\boldsymbol{s}^+_\nu \times \tilde{\boldsymbol{d}}_\nu]^{(2)})]^{(0)} + \\
&\chi_\nu\,\chi_\pi\,[[\boldsymbol{d}^+_\pi \times \tilde{\boldsymbol{d}}_\pi]^{(2)} \times [\boldsymbol{d}^+_\nu \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)}\}.
\end{aligned}$$

The last term in equation (15.16) is named Majorana term and it reads

$$\begin{aligned}
\boldsymbol{M}_{\pi\nu} = {}&\sqrt{5}\{([[\boldsymbol{d}^+_\pi \times \boldsymbol{s}^+_\nu]^{(2)} \times [\tilde{\boldsymbol{d}}_\pi \times \boldsymbol{s}_\nu]^{(2)}]^{(0)} + [[\boldsymbol{s}^+_\pi \times \boldsymbol{d}^+_\nu]^{(2)} \times [\boldsymbol{s}_\pi \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)}) - \\
&([[\boldsymbol{d}^+_\pi \times \boldsymbol{s}^+_\nu]^{(2)} \times [\boldsymbol{s}_\pi \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)} + [[\boldsymbol{s}^+_\pi \times \boldsymbol{d}^+_\nu]^{(2)} \times [\tilde{\boldsymbol{d}}_\pi \times \boldsymbol{s}_\nu]^{(2)}]^{(0)})\} + \\
&2\,\textstyle\sum_{J=1,3}\sqrt{(2J+1)}\,[[\boldsymbol{d}^+_\pi \times \boldsymbol{d}^+_\nu]^{(J)} \times [\tilde{\boldsymbol{d}}_\pi \times \tilde{\boldsymbol{d}}_\nu]^{(J)}]^{(0)}. \qquad (15.20)
\end{aligned}$$

Sometimes a further parameter is attached to the last expression. The terms in the expressions (15.19) and (15.20) appear individually in the original expression for $\boldsymbol{V}_{\pi\nu}$ (15.7), which can be pursued by means of table15.1.

Table 15.1

Correspondence of terms in (15.7), (15.9) and (15.10) to the terms in (15.19) and (15.20).

Original terms of $\boldsymbol{V}_{\pi\nu}$ in (15.7), (15.9) and (15.10)	corresponding terms in (15.19) and (15.20), the numbers denote the lines.
Term with W_9 (see 15.9)	1. Term in $\sqrt{5}\,[\boldsymbol{Q}_\pi(\chi_\pi) \times \boldsymbol{Q}_\nu(\chi_\nu)]^{(0)}$, (15.19)
" " W_7	2. " " ",
" " W_{10}	3. " " ",
" " W_{11}	4. " " ",
Terms with W_0 .. W_4 (see 15.10)	5. " " ",
" " W_6 and W_8	1. Term in $\boldsymbol{M}_{\pi\nu}$ (15.20),
Term with W_7	2. " " ",
Terms with W_1 and W_3	3. " " ".

This comparison shows that in fact only the coefficients of the terms in $\boldsymbol{V}_{\pi\nu}$ have been altered. In doing so, individual terms had to be recoupled by means of (A4.1). The consistent correspondence reveals that expression (15.16) is not an ad hoc construction but the result of a reduction and remodelling of the operator (15.7).

The quadrupole-quadrupole operator has to replace to a certain degree the operators $\boldsymbol{Q}^2_\pi$ and $\boldsymbol{Q}^2_\nu$, which have been omitted in $\boldsymbol{H}_\pi$ and $\boldsymbol{H}_\nu$ (15.15). Furthermore,one knows from the shell model that the strongly attractive interaction between proton and neutron breaks the so-called generalised seniority (Talmi, 1993, p. 906) and is responsible for the deformation of the nucleus (Eisenberg and Greiner, 1987, p. 443). This π-ν-interaction corresponds to the *sd*-terms in $\boldsymbol{Q}_\rho(\chi_\rho)_\mu$ (15.17), which influence the *s*-*d*-division of bosons.

In order to illustrate this behaviour we show that the term $\boldsymbol{Q}_\pi(\chi_\pi)\bullet\boldsymbol{Q}_\nu(\chi_\nu)$ (15.18, 15.19) is able to make up non vanishing matrix elements resting on the states $\boldsymbol{s}^+_\pi\,\boldsymbol{s}^+_\nu\,|\rangle$ and $[\boldsymbol{d}^+_\pi \times \boldsymbol{d}^+_\nu]^{(0)}|\rangle$. We pursue this by means of the first term in (15.19), $[[\boldsymbol{d}^+_\pi \times \boldsymbol{s}_\pi]^{(2)} \times [\boldsymbol{d}^+_\nu \times \boldsymbol{s}_\nu]^{(2)}]^{(0)}$, as follows

$$\begin{aligned}
&\langle|\,[\tilde{\boldsymbol{d}}_\pi \times \tilde{\boldsymbol{d}}_\nu]^{(0)}\,[[\boldsymbol{d}^+_\pi \times \boldsymbol{s}_\pi]^{(2)} \times [\boldsymbol{d}^+_\nu \times \boldsymbol{s}_\nu]^{(2)}]^{(0)}\,\boldsymbol{s}^+_\pi \boldsymbol{s}^+_\nu|\rangle = \\
&\langle|\,[\tilde{\boldsymbol{d}}_\pi \times \tilde{\boldsymbol{d}}_\nu]^{(0)}\,[\boldsymbol{d}^+_\pi \times \boldsymbol{d}^+_\nu]^{(0)}\,\boldsymbol{s}_\pi \boldsymbol{s}_\nu \boldsymbol{s}^+_\pi \boldsymbol{s}^+_\nu|\rangle = \qquad (15.21)\\
&\langle|\,[\boldsymbol{d}_\pi \times \boldsymbol{d}_\nu]^{(0)}\,[\boldsymbol{d}^+_\pi \times \boldsymbol{d}^+_\nu]^{(0)}\,|\rangle = 1.
\end{aligned}$$

Thus, the quadrupole-quadrupole operator creates non-diagonal matrix elements, which cause s-*d*-mixing.

The Majorana term $\boldsymbol{M}_{\pi\nu}$ (15.20) influences states with a special symmetry which is forbidden in the IBM1. We show that by means of states with $N_\pi = N_\nu = 1$ (15.13), which can be remodelled this way

$$|[s_\pi \times s_\nu]^{(0)}\rangle,\ (|[s_\pi \times d_\nu]^{(2)}{}_\mu\rangle + |[d_\pi \times s_\nu]^{(2)}{}_\mu\rangle)/\sqrt{2},\ |[d_\pi \times d_\nu]^{(J)}{}_{\mu'}\rangle,\ J \text{ even}, \qquad (15.22)$$

$$(|[s_\pi \times d_\nu]^{(2)}{}_\mu\rangle - |[d_\pi \times s_\nu]^{(2)}{}_\mu\rangle)/\sqrt{2},\ |[d_\pi \times d_\nu]^{(J)}{}_{\mu'}\rangle,\ J \text{ odd}. \qquad (15.23)$$

The states (15.22) are symmetric with regard to the indices π and ν and the states (15.23) are antisymmetric. If we replace the labels π and ν by boson numbers 1 and 2, equation (15.22) represents the symmetric two-boson states of the IBM1 but the states (15.23) exist neither in this model nor in reality. By means of the Majorana term we can lift these intruder states in a higher energy than all decisive levels and make them insignificant.

For example, we consider first the influence of $\boldsymbol{M}_{\pi\nu}$ on one partial state in (15.23)

$$\begin{aligned}
&(1/\sqrt{5})\,\boldsymbol{M}_{\pi\nu}|\,[s_\pi \times d_\nu]^{(2)}{}_\mu\,\rangle = \\
&[[\boldsymbol{s}^+_\pi \times \boldsymbol{d}^+_\nu]^{(2)} \times [\boldsymbol{s}_\pi \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)}\,\boldsymbol{s}^+_\pi \boldsymbol{d}^+_{\nu\mu}\,|\rangle - \qquad (15.24)\\
&[[\boldsymbol{d}^+_\pi \times \boldsymbol{s}^+_\nu]^{(2)} \times [\boldsymbol{s}_\pi \times \tilde{\boldsymbol{d}}_\nu]^{(2)}]^{(0)}\,\boldsymbol{s}^+_\pi \boldsymbol{d}^+_{\nu\mu}\,|\rangle.
\end{aligned}$$

All other states vanish because each time an annihilation operator acts on an empty state $|\rangle$. We remodel the expression (15.24)

$$(1/\sqrt{5})\, \boldsymbol{M_{\pi\nu}} |\, [s_{\,\pi} \times d_\nu]^{(2)}{}_\mu \rangle =$$
$$\textstyle\sum_\lambda \boldsymbol{d^+_{\nu\lambda} d^{\sim}_{\nu,-\lambda}} (2\ \lambda\ 2, -\lambda \mid 0\ 0)\, \boldsymbol{d^+_{\nu\mu} s^+_\pi s_\pi s^+_\pi} |\rangle\ - \qquad (15.25)$$
$$\textstyle\sum_\lambda \boldsymbol{d^+_{\pi\lambda} d^{\sim}_{\nu,-\lambda}} (2\ \lambda\ 2, -\lambda \mid 0\ 0)\, \boldsymbol{d^+_{\nu\mu} s^+_\nu s_\pi s^+_\pi} |\rangle.$$

Because of $\boldsymbol{d^{\sim}_{-\lambda}} = \boldsymbol{d_\lambda} (-1)^{-\lambda}$ and $(2\ \lambda\ 2, -\lambda \mid 0\ 0) = (-1)^\lambda/\sqrt{5}$ the relation

$$(1/\sqrt{5})\, \boldsymbol{M_{\pi\nu}} |\, [s_{\,\pi} \times d_\nu]^{(2)}{}_\mu \rangle = (1/\sqrt{5})\, (\boldsymbol{d^+_{\nu\mu} s^+_\pi} - \boldsymbol{d^+_{\pi\mu} s^+_\nu}) |\rangle \text{ holds.} \qquad (15.26)$$

Analogously one gets

$$\boldsymbol{M_{\pi\nu}} |\, [d_{\,\pi} \times s_\nu]^{(2)}{}_\mu \rangle = (\boldsymbol{d^+_{\pi\mu} s^+_\nu} - \boldsymbol{d^+_{\nu\mu} s^+_\pi}) |\rangle.$$

Therefore, for corresponding states of (15.22) and (15.23) we obtain

$$\boldsymbol{M_{\pi\nu}} (|[s_\pi \times d_\nu]^{(2)}{}_\mu\rangle + |[d_\pi \times s_\nu]^{(2)}{}_\mu\rangle)/\sqrt{2} = 0, \qquad (15.27)$$
$$\boldsymbol{M_{\pi\nu}} (|[s_\pi \times d_\nu]^{(2)}{}_\mu\rangle - |[d_\pi \times s_\nu]^{(2)}{}_\mu\rangle)/\sqrt{2} =$$
$$\sqrt{2}(|[s_\pi \times d_\nu]^{(2)}{}_\mu\rangle - |[d_\pi \times s_\nu]^{(2)}{}_\mu\rangle). \qquad (15.28)$$

Now we investigate the effect of $\boldsymbol{M_{\pi\nu}}$ on other states of (15.22) and (15.23) like this

$$\sqrt{(2J+1)}\, [[\boldsymbol{d^+_\pi} \times \boldsymbol{d^+_\nu}]^{(J)} \times [\boldsymbol{d^{\sim}_\pi} \times \boldsymbol{d^{\sim}_\nu}]^{(J)}]^{(0)}\, |[d_\pi \times d_\nu]^{(L)}{}_{M_L}\rangle =$$
$$\textstyle\sum_M (-1)^{J-M} [\boldsymbol{d^+_\pi} \times \boldsymbol{d^+_\nu}]^{(J)}{}_M\, [\boldsymbol{d^{\sim}_\pi} \times \boldsymbol{d^{\sim}_\nu}]^{(J)}{}_{-M}\, (1/\sqrt{2}) [\boldsymbol{d_\pi} \times \boldsymbol{d_\nu}]^{(L)}{}_{M_L} |\rangle = \qquad (15.29)$$
$$(1/\sqrt{2}) \textstyle\sum_M (-1)^{J-M} [\boldsymbol{d^+_\pi} \times \boldsymbol{d^+_\nu}]^{(J)}{}_M \cdot$$
$$\textstyle\sum_{\lambda_\pi \lambda_\nu \mu_\pi \mu_\nu} (2\ \lambda_\pi\ 2\ \lambda_\nu \mid J, -M)(2\ \mu_\pi\ 2\ \mu_\nu \mid L\ M_L)(-1)^{\lambda_\pi + \lambda_\nu} \boldsymbol{d_{\pi,-\lambda_\pi} d_{\nu,-\lambda_\nu} d^+_{\pi\,\mu_\pi} d^+_{\nu\,\mu_\nu}} |\rangle =$$
$$(1/\sqrt{2}) \textstyle\sum_M (-1)^J\, [\boldsymbol{d^+_\pi} \times \boldsymbol{d^+_\nu}]^{(J)}{}_M \cdot \sum_{\mu_\pi \mu_\nu} (2, -\mu_\pi\ 2, -\mu_\nu | J, -M)(2\ \mu_\pi\ 2\ \mu_\nu | L\ M_L) |\rangle =$$
$$(1/\sqrt{2}) \textstyle\sum_M [\boldsymbol{d^+_\pi} \times \boldsymbol{d^+_\nu}]^{(J)}{}_M \cdot \sum_{\mu_\pi \mu_\nu} (2\ \mu_\pi\ 2\ \mu_\nu \mid J, -M)(2\ \mu_\pi\ 2\ \mu_\nu \mid L\ M_L) |\rangle =$$
$$(1/\sqrt{2}) [\boldsymbol{d^+_\pi} \times \boldsymbol{d^+_\nu}]^{(L)}{}_{M_L} |\rangle\ \delta_{JL}.$$

Therefore, we obtain

$$\boldsymbol{M_{\pi\nu}}\, |[d_\pi \times d_\nu]^{(L)}{}_{M_L}\rangle = +\sqrt{2} [\boldsymbol{d^+_\pi} \times \boldsymbol{d^+_\nu}]^{(L)}{}_{M_L} |\rangle\ (\delta_{1L} + \delta_{3L}). \qquad (15.30)$$

For the first expression in (15.22) the relation

$$\boldsymbol{M_{\pi\nu}}\, |s_\pi s_\nu\rangle = 0 \qquad (15.31)$$

holds. We see how the Majorana operator $\boldsymbol{M_{\pi\nu}}$ contributes to diagonal elements of the Hamilton matrix on the basis (15.22) and (15.23). According to (15.27), (15.28), (15.30) and (15.31) only the diagonal matrix elements of the antisymmetric states (15.23) are non-vanishing and positive if the coefficient λ in (15.16) is positive. By means of λ one lifts the levels of the states which are antisymmetric with respect to π and ν. Doing so, they lose all spectroscopic significance. One can show that $\boldsymbol{M_{\pi\nu}}$ acting on more complex basis states has the same effect.

15.4 Eigenenergies and electromagnetic transitions within the IBM2

Having discussed the individual terms of the IBM2-Hamiltonian (see 15.15 up to 15.20) we put it together

$$\boldsymbol{H} = \varepsilon_\pi\, \boldsymbol{n_{d\,\pi}} + \varepsilon_\nu\, \boldsymbol{n_{d\,\nu}} +$$

$$\sum_{\rho=\pi,\,\nu} \tfrac{1}{2} \sum_{J=0,\,2,\,4} c_{J\,\rho} \sqrt{(2J+1)}\, [[\boldsymbol{d}^{+}_{\rho} \times \boldsymbol{d}^{+}_{\rho}]^{(J)} \times [\boldsymbol{d}^{\sim}_{\rho} \times \boldsymbol{d}^{\sim}_{\rho}]^{(J)}]^{(0)} +$$

$$\kappa \sqrt{5}\, [\boldsymbol{Q_\pi}(\chi_\pi) \times \boldsymbol{Q_\nu}(\chi_\nu)]^{(0)} + \lambda\, \boldsymbol{M_{\pi\nu}}. \qquad (15.32)$$

It contains 12 parameters. The eigenenergies of $\boldsymbol{H}$ are calculated on a basis of functions analogous to (12.2) which are composed of the basis states (15.12) like this

$$|\, I,\, N_\pi\, N_\nu\;\, J\, M\,\rangle\rangle = \sum_i\; c^{(I)}{}_{N_\pi N_\nu\, J}\,(i)\;|\; i,\, N_\pi\, N_\nu\;\, J\, M\,\rangle. \qquad (15.33)$$

The coefficients $c^{(I\;)}{}_{N_\pi\;N_\nu\;J}\;(i)$ of the Ith state result from the diagonalisation procedure discussed in section 12.1. The sum comprises the configurations i = ($n_{s\pi}\;\; n_{d\pi}\;\; \tau_\pi\;\; n_{\Delta\pi}\;\; J_\pi\;\; n_{s\nu}\;\; n_{d\nu}\;\; \tau_\nu\;\; n_{\Delta\nu}\;\; J_\nu$) according to (15.12). The eigenvalue is labelled with $E^{(I\;)}{}_{N_\pi\,N_\nu\,J}$.

The probabilities for electromagnetic transitions between states of this kind are formulated analogously to the method described in chapter 11. The reduced transition probability $B(L{,}J_i \rightarrow J_f\,)$ is calculated by means of the initial and the final state of the type (15.33). For the operator $\boldsymbol{O}$(E L M) for an electromagnetic transition with the angular momentum L, the form (11.4) is extended to a sum of the operator for π-bosons and the one for ν-bosons.

Specially for electric transitions with L = 2 (quadrupole radiation) we have analogously to (11.12) the following expression

$$\boldsymbol{O}(\mathrm{E}\;2\;\mu) = \sum_{\rho=\pi,\nu} e_\rho\, (\boldsymbol{d}^{+}_{\rho\mu}\boldsymbol{s_\rho} + \boldsymbol{s}^{+}_{\rho}\boldsymbol{d}^{\sim}_{\rho\,\mu} + \chi'_\rho\, [\boldsymbol{d}^{+}_{\rho} \times \boldsymbol{d}^{\sim}_{\rho}]^{(2)}{}_{\mu}). \qquad (15.34)$$

The operators in (15.34) correspond largely to the operator $\boldsymbol{Q_\rho}(\chi_\rho)_\mu$ in (15.17), for which reason frequently one puts $\chi'_\rho = \chi_\rho$. This demands that this parameter is fitted simultaneously on energies and on transition probabilities. In this case, we have

$$\boldsymbol{O}(\mathrm{E}\;2\;\mu) = e_\pi\, \boldsymbol{Q_\pi}(\chi_\pi)_\mu + e_\nu\, \boldsymbol{Q_\nu}(\chi_\nu)_\mu\,. \qquad (15.35)$$

Frequently the quantity e_π is equated with e_ν. T. Otsuka and O. Scholten have written the computer program NPBOS which performs IBM2-calculations essentially according to (15.32) (T. Otsuka, 1977, University Tokyo). The newer program BOSON2 (A. Novoselsky, 1985, Weizmann Institute, Israel) is able to handle with a great number of bosons.

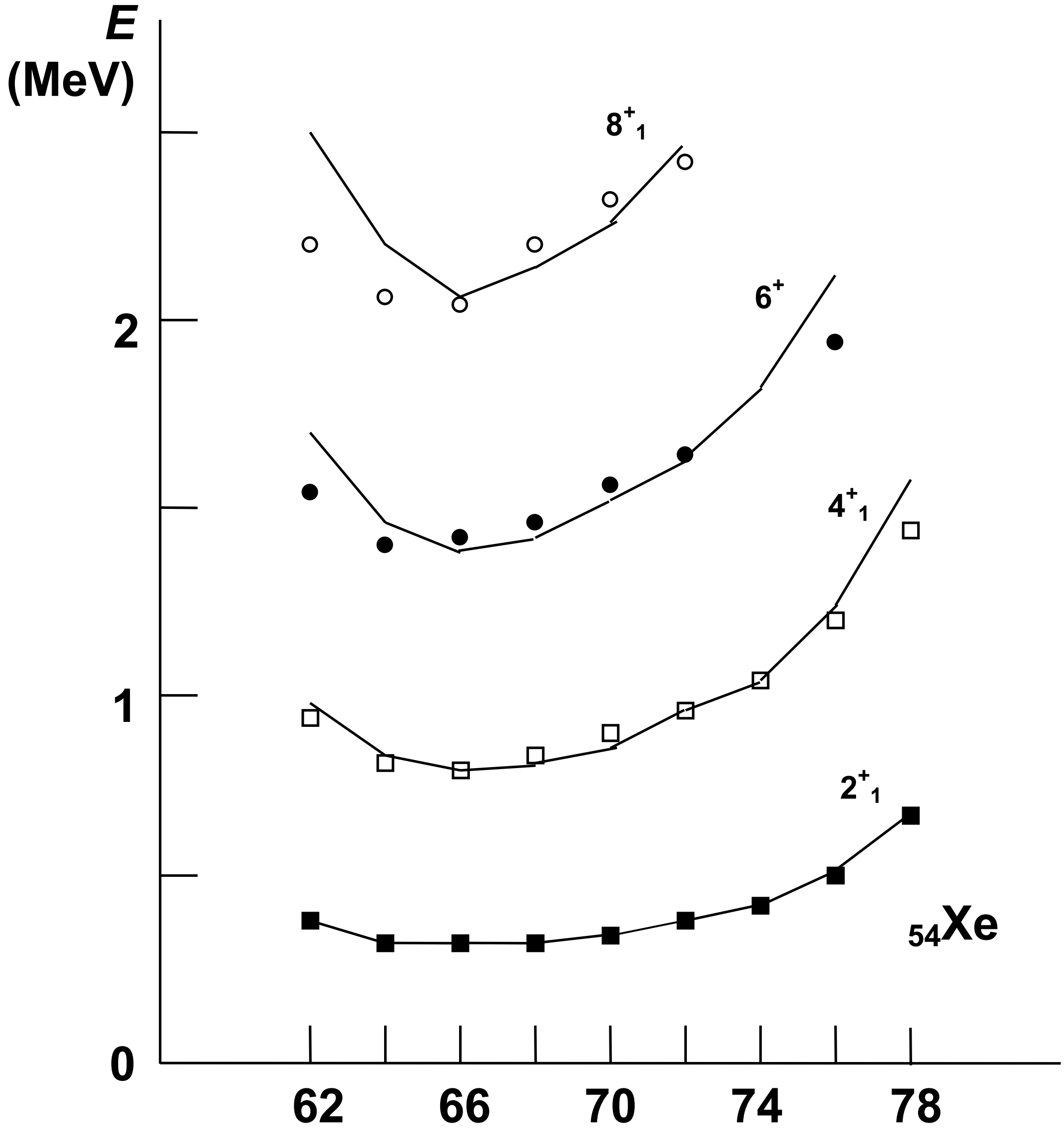

Figure 15.1. Comparison between the measured and the calculated energies of the levels 2^+_1, 4^+_1, 6^+_1 and 8^+_1 of isotopes of $_{54}$Xe with even numbers of neutrons 62 up to 78. The IBM2-Hamiltonian has been employed (Scholten, 1980, p. 55).

15.5 Comparisons with experimental data

Figures 15.1 up to 15.3 show experimental and calculated energies of levels of $_{54}$Xe and $_{56}$Ba isotopes. They are taken from the report of O. Scholten (1980). The Hamiltonian (15.32) has been employed with the simplification $e_\pi = e_\nu$. The Majorana term $\boldsymbol{M}_{\pi\nu}$ has been formulated more generally.

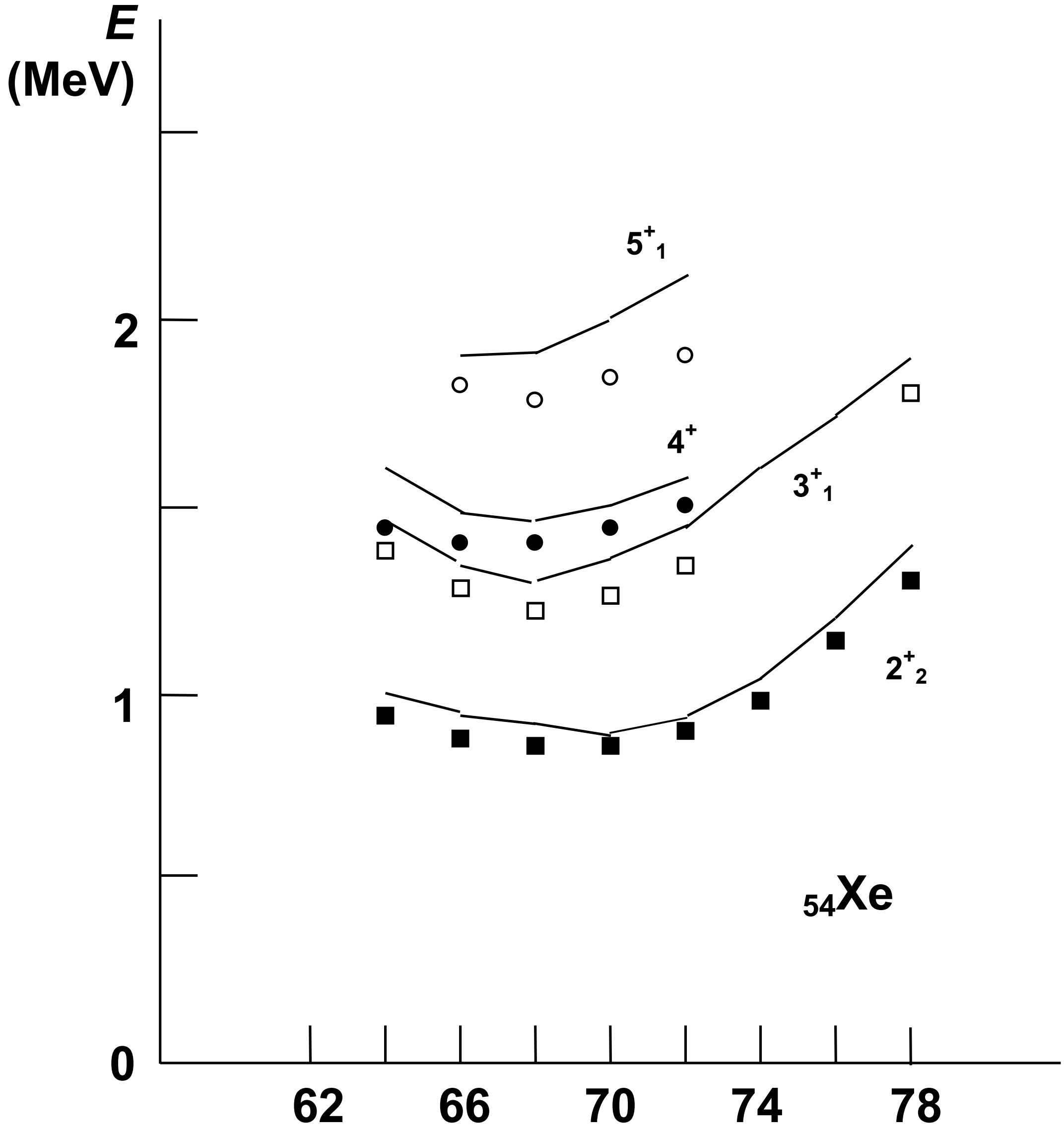

Figure 15.2. Comparison between the measured and the calculated energies of the levels 5^+_1, 4^+_2, 3^+_1 and 2^+_2 of isotopes of $_{54}$Xe with even numbers of neutrons 64 up to 78 (Scholten, 1980, p. 55).

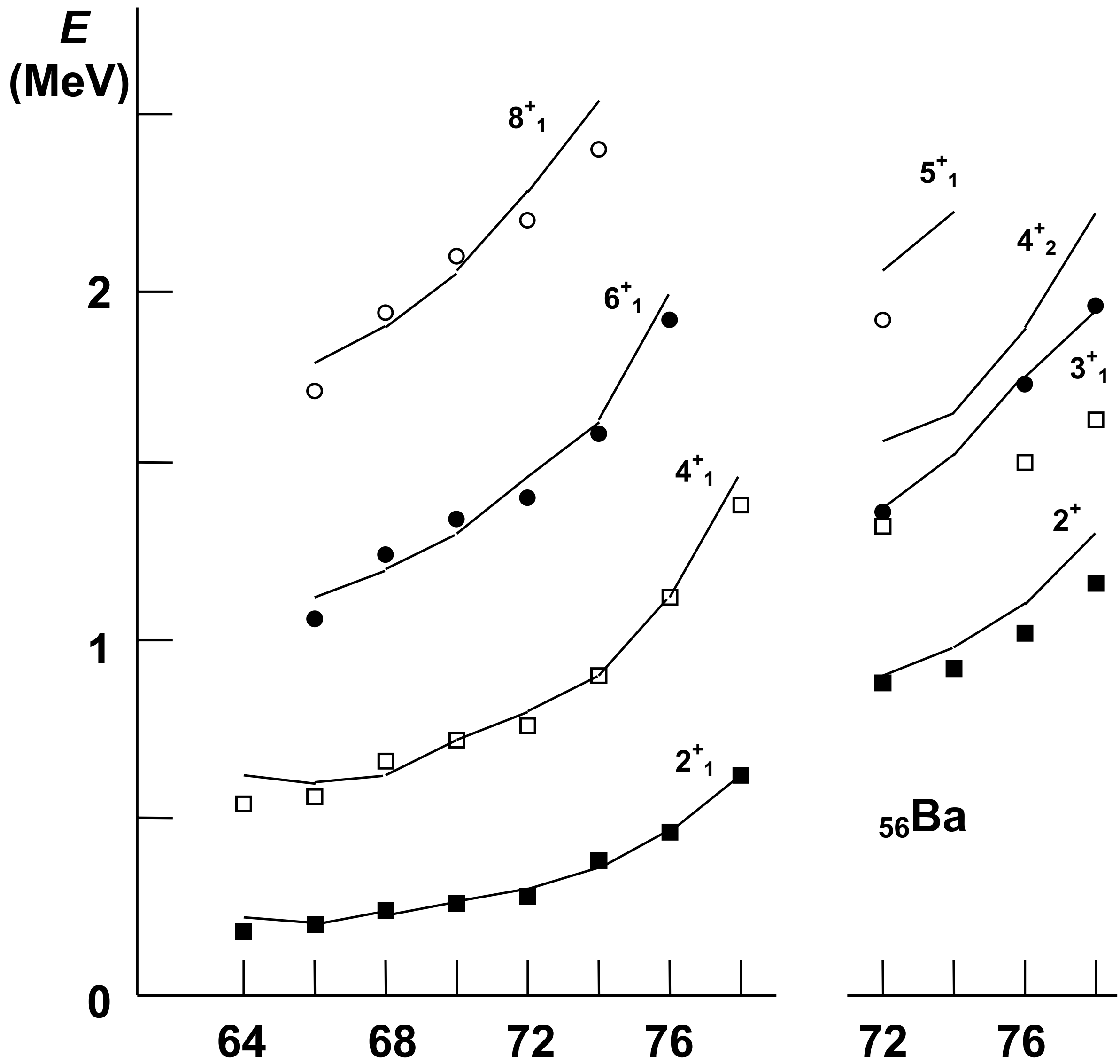

Figure 15.3. Comparison between the measured and the calculated energies of 8 levels of $_{56}$Ba isotopes with even numbers of neutrons 64 up to 78. The IBM2-Hamiltonian has been employed (Scholten, 1980, p. 55).

The corresponding parameters ε and κ of the Hamilton operator are represented in figure 15.4 as a function of the neutron number. The dependence of these quantities on the proton number (54 up to 58) could be neglected.

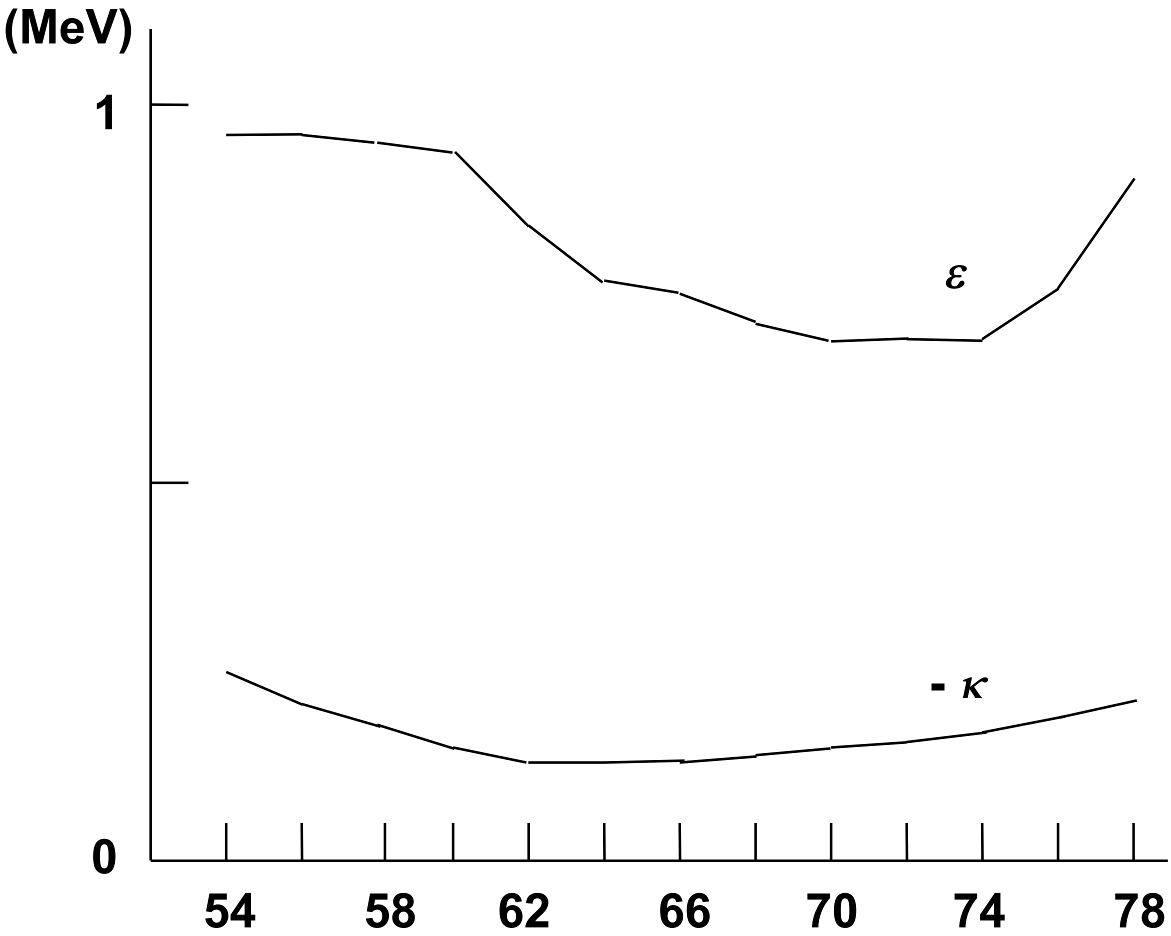

Figure 15.4. The parameters κ (with negative values) and ε employed for the IBM2-calculations shown in figures 15.1 up to 15.3 for nuclei with even neutron numbers from 54 up to 78 and proton numbers 54 up to 58 (Scholten, 1980, p. 57).

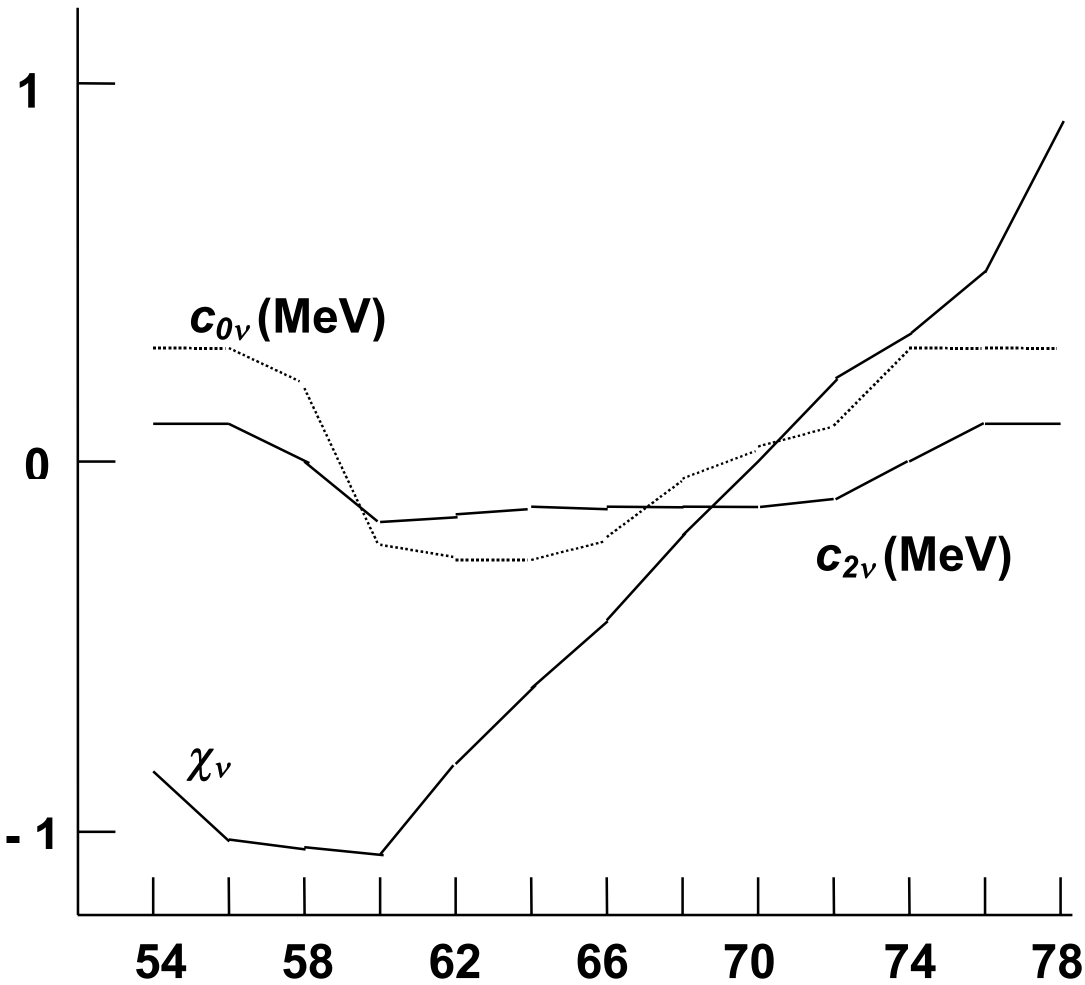

Figure 15.5. The parameters χ_ν , $c_{0\nu}$ and $c_{2\nu}$ employed for the IBM2-calculations shown in figures 15.1 up to 15.3 for nuclei with even neutron numbers 54 up to 78 (Scholten, 1980, p. 57).

Figure 15.5 shows the separate dependence of the quantities χ_ν, $c_{0\nu}$ and $c_{2\nu}$ on the neutron number. The coefficient $c_{4\nu}$ was put equal to zero. Also the quantities $c_{0\pi}$, $c_{2\pi}$ and $c_{4\pi}$ are nearly zero, but χ_π descends from -0.81 to -1 for proton numbers 54 up to 58. Furthermore, O. Scholten (1980) showed that the terms of $c_{0\nu}$, $c_{2\nu}$ and $c_{4\nu}$ have little influence. Compared with the quadrupole term $\kappa\sqrt{5}[\boldsymbol{Q}_\pi(\chi_\pi) \times \boldsymbol{Q}_\nu(\chi_\nu)]^{(0)}$ they are only important if there are little neutrons beside many active protons or vice versa.

Reduced transition probabilities for quadrupole radiation are calculated according to (15.35) and shown in figure 15.6 together with experimental data.

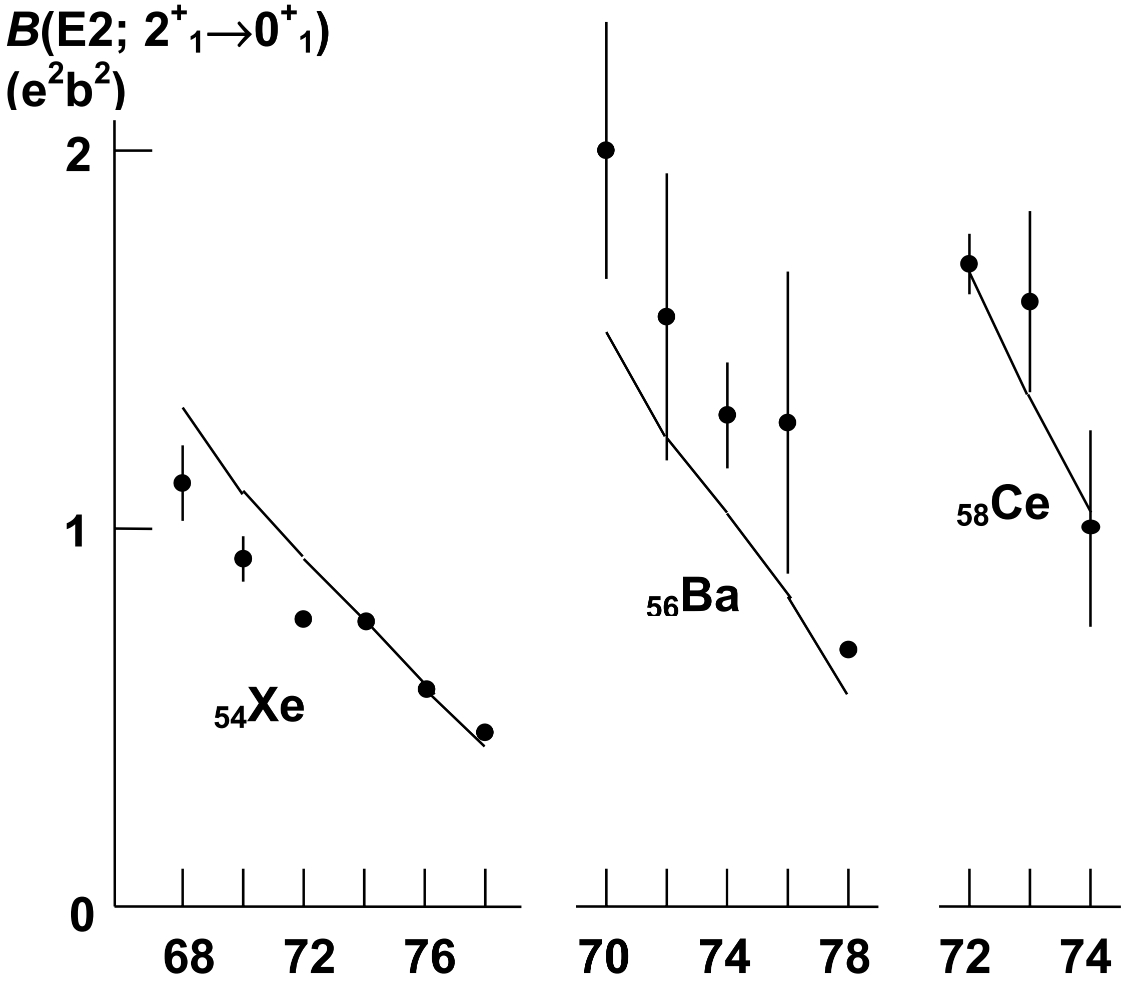

Figure 15.6. Comparison between measured and calculated reduced transition probabilities for quadrupole radiation of even-even nuclei with neutron numbers 68 up to 78. The parameters of figures 15.4 and 15.5 have been employed (Scholten, 1980, p. 58).

Duval and Barrett (1981) have performed similar IBM2-calculations for tungsten isotopes. They employed the parameters $\varepsilon_\pi = \varepsilon_\nu$ and $c_{4\nu} = c_{0\pi} = c_{2\pi} = c_{4\pi} = 0$. Figures 15.7 and 15.8 show the comparison with experimental data.

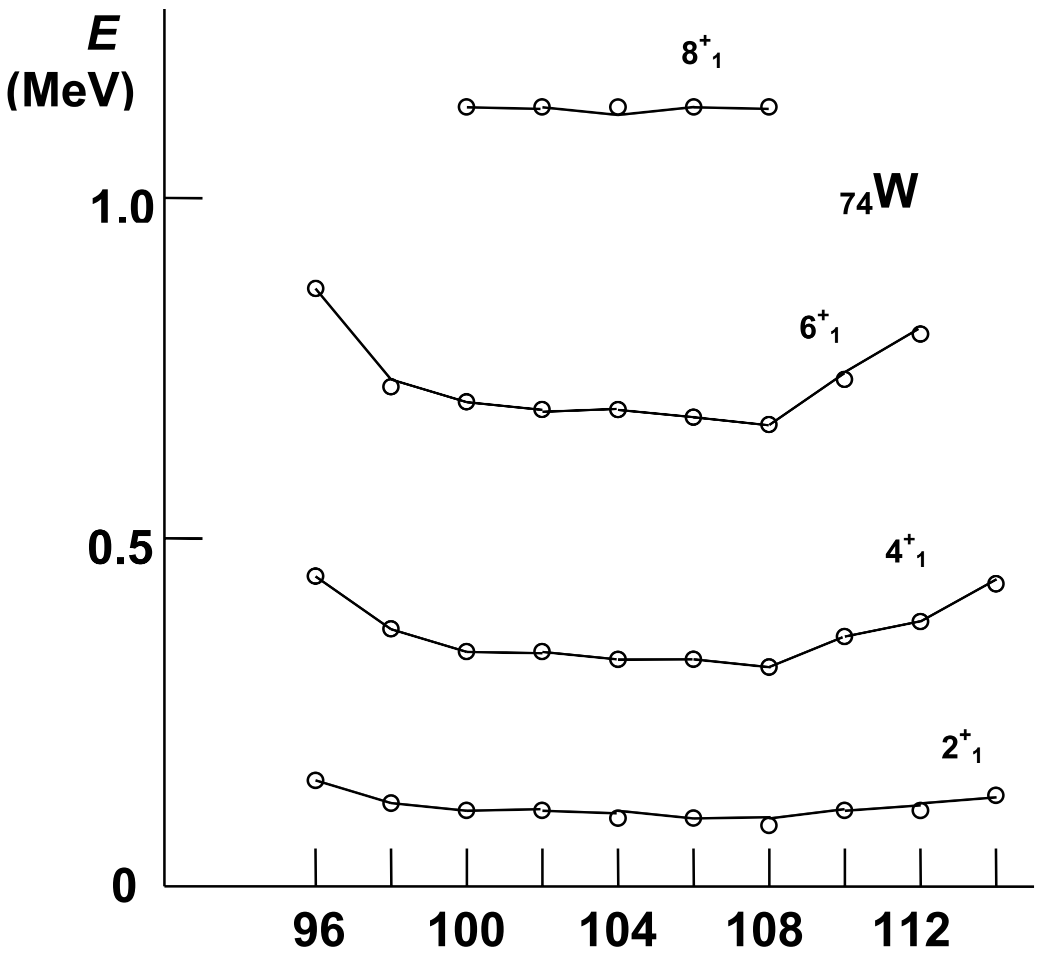

Figure 15.7. Comparison between the measured and calculated energies of the levels 2^+_1, 4^+_1, 6^+_1 and 8^+_1 of isotopes from $_{74}W$ with even neutron numbers 96 up to 114 employing the IBM2 formalism (Duval and Barrett, 1981, p. 493).

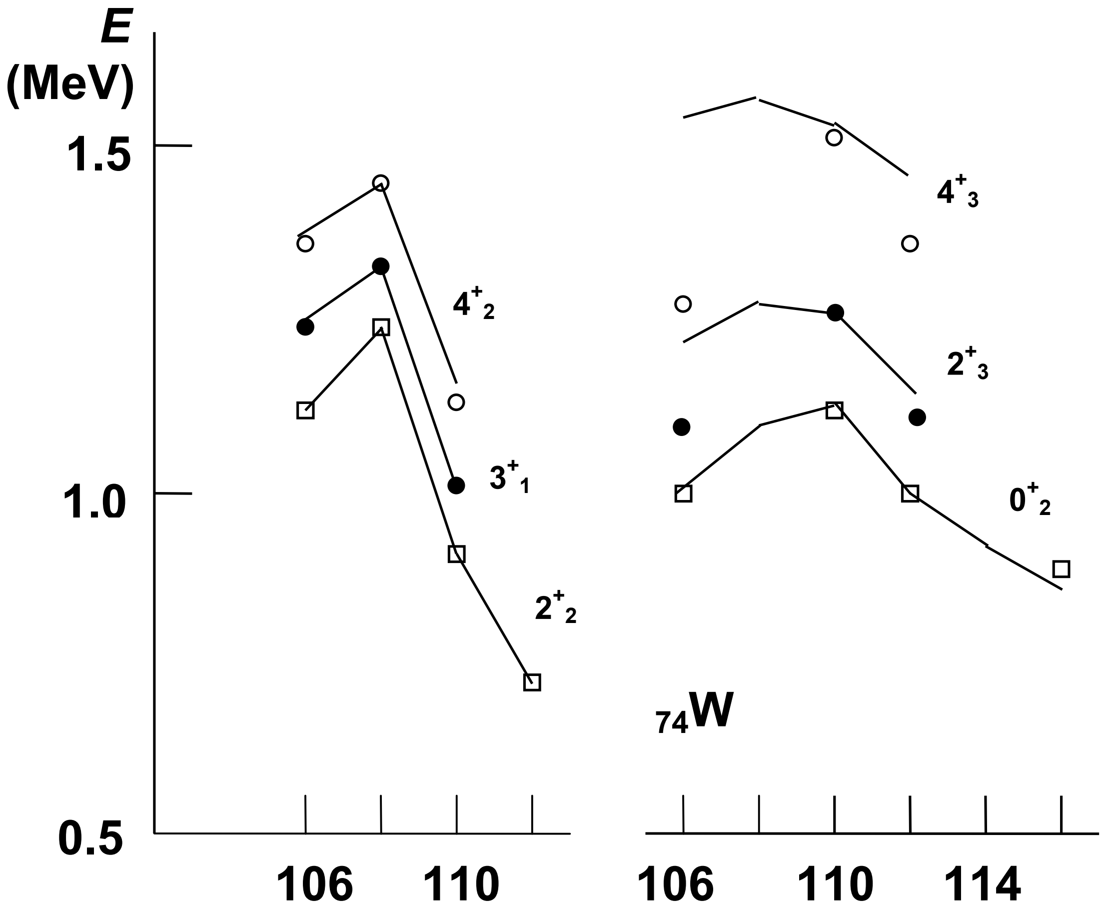

Figure 15.8. Comparison between measured and calculated energies of 6 levels of $_{74}$W with even numbers of neutrons 106 up to 116. The IBM2-Hamiltonian has been employed (Duval and Barrett, 1981, p. 494).

In figures 15.9 and 15.10, the behaviour of the corresponding IBM2-parameters depending on the neutron number is shown.

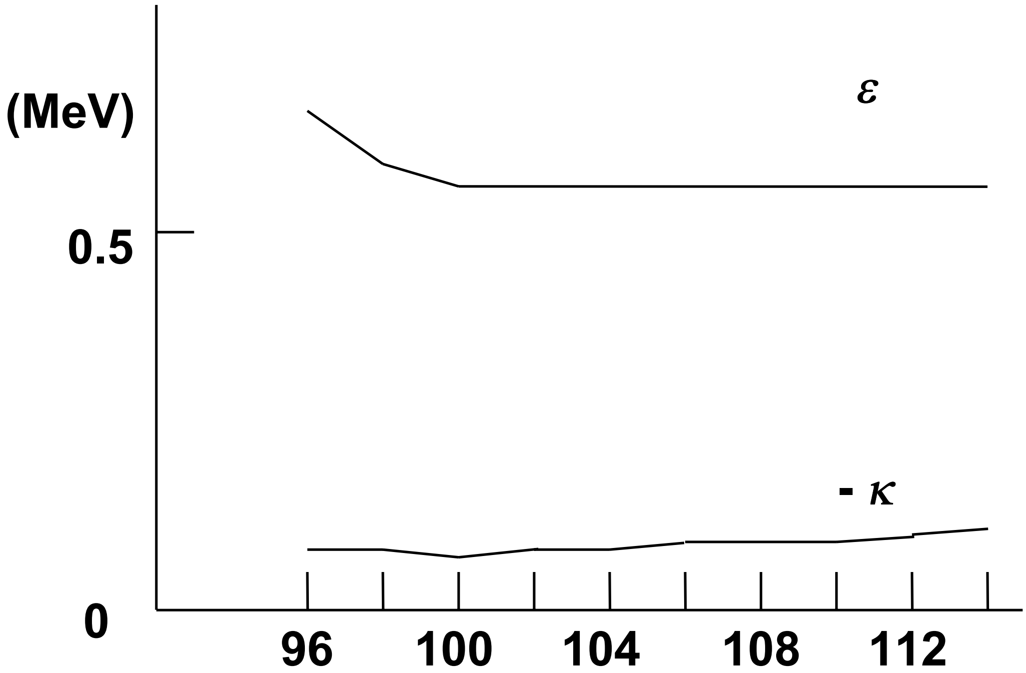

Figure 15.9. The parameters κ (with negative values) and ε employed for the IBM2-calculations shown in figures 15.7 and 15.8 for tungsten isotopes with even neutron numbers 96 up to 114 (Duval and Barrett, 1981, p. 495).

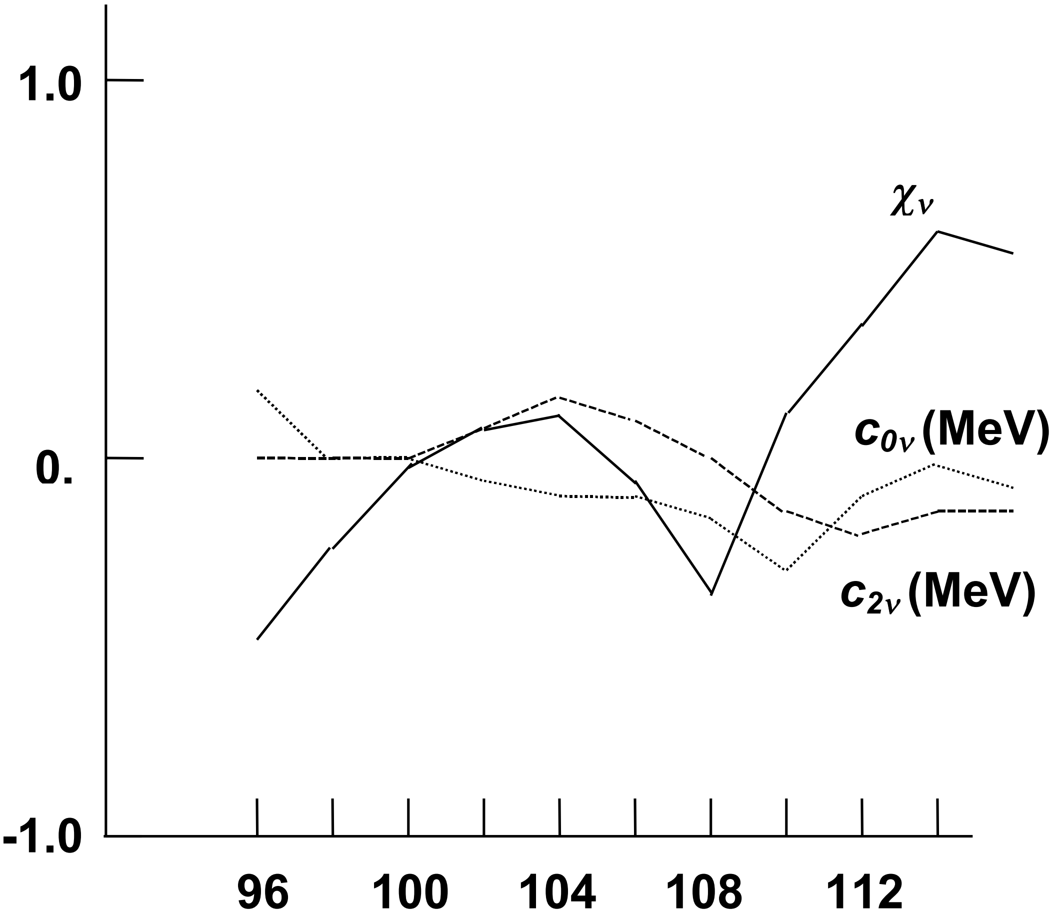

Figure 15.10. The parameters χ_ν , $c_{0\nu}$ and $c_{2\nu}$ employed for the IBM2-calculations shown in figures 15.7 and 15.8 for tungsten isotopes with even neutron numbers 96 up to 114 (Duval and Barrett, 1981, p. 495).

Figure 15.11 shows some reduced transition probabilities B(E2) for γ -radiation and compares calculated with measured values. Jolos et al. (1985) have performed similar calculations for isotopes of $_{44}$Ru and $_{46}$Pd with equal success.

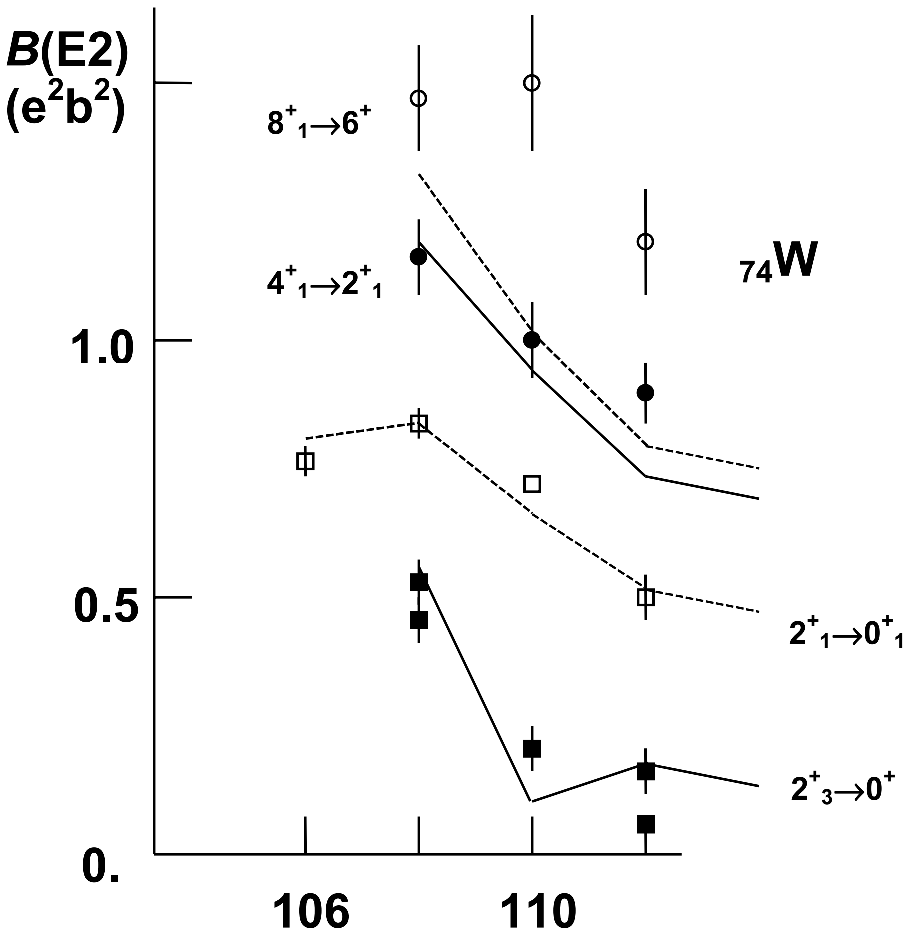

Figure 15.11. Comparison between measured and calculated reduced transition probabilities for quadrupole radiation of tungsten isotopes with neutron numbers 106 up to 112 (Duval and Barrett, 1981, p. 495 up to p. 497). The employed parameters are shown in figures 15.9 and 15.10 and described in the text.

As we have mentioned above (Scholten, 1980) little changes if all coefficients $c_{J\rho}$ in (15.32) are equal to zero and $\varepsilon_\pi = \varepsilon_\nu$. In this case the Hamilton operator obtains the following form

$$\boldsymbol{H} = \varepsilon\,(\boldsymbol{n}_{d\,\pi} + \boldsymbol{n}_{d\,\nu}) + \kappa\,\sqrt{5}\,[\boldsymbol{Q}_\pi(\chi_\pi) \times \boldsymbol{Q}_\nu(\chi_\nu)]^{(0)} + \lambda\,\boldsymbol{M}_{\pi\nu}. \qquad (15.36)$$

With this Hamiltonian, which contains essentially four parameters (frequently λ is fixed arbitrarily), for many nuclei good agreement could be achieved with measured data. Figure 0.1 puts together existing calculations up to 1988. The expression (15.36) is named Talmi's Hamiltonian (Iachello, Arima, 1987, p.140). In the same publication on page 140 a group of authors is given which have done calculations with this Hamilton operator.

15.6 Chains of Lie algebras in the IBM2

Owing to the relationship between the IBM2- and the IBM1-Hamiltonian we expect also chains of Lie algebras such as in (14.11), (14.17), (14.21) and table 14.1.

First, we consider the term $\boldsymbol{V_{\pi\nu}}$ (15.7). It contains a sum comprising expressions of the following form

$$[[\boldsymbol{b}^{+}_{\pi j_f} \times \boldsymbol{b}^{+}_{\nu j_g}]^{(J)} \times [\boldsymbol{b}^{\sim}_{\pi j_p} \times \boldsymbol{b}^{\sim}_{\nu j_q}]^{(J)}]^{(0)}\ ,\ (j_f, j_g, j_p, j_q = 0{,}2\ ) \tag{15.37}$$

where the labels π and ν are ordered. With the aid of the 9-j symbols, (A4.1) one obtains sums of the following expressions

$$[[\boldsymbol{b}^{+}_{\pi j_f} \times \boldsymbol{b}^{\sim}_{\pi j_p}]^{(J)} \times [\boldsymbol{b}^{+}_{\nu j_g} \times \boldsymbol{b}^{\sim}_{\nu j_q}]^{(J)}]^{(0)}\ . \tag{15.38}$$

On the other hand the terms $\boldsymbol{H_\rho}$ ($\rho = \pi,\ \nu$) in (15.4) consist according to (14.2), (14.3), (14.5) and (14.7) on sums of expressions such as

$$[\boldsymbol{b}^{+}_{\rho j_1} \times \boldsymbol{b}^{\sim}_{\rho j_2}]^{(0)} \text{ and } [[\boldsymbol{b}^{+}_{\rho j_1} \times \boldsymbol{b}^{\sim}_{\rho j_2}]^{(J)} \times [\boldsymbol{b}^{+}_{\rho j_3} \times \boldsymbol{b}^{\sim}_{\rho j_4}]^{(J)}]^{(0)}. \tag{15.39}$$

We single out special cases of Hamilton operators (15.4) which are able to constitute chains of Lie algebras.

First series of special cases of the IBM2

Here all corresponding coefficients in $\boldsymbol{H_\pi}$ and $\boldsymbol{H_\nu}$ (14.2) are put equal i. e.

$$\varepsilon_{\pi\, n} = \varepsilon_{\nu\, \mathrm{n}} \equiv \varepsilon_n,\ \ \ldots,\ v_{\pi\, n} = v_{\nu\, n} \equiv v_n,\ \ \ldots,\ v_{\pi\, q} = v_{\nu\, q} \equiv v_q\ . \tag{15.40}$$

If one replaces in $\boldsymbol{H}$ (14.2) (14.3), (14.5) and (14.7) every term of the form

$$[\boldsymbol{b}^{+}_{j_a} \times \boldsymbol{b}^{\sim}_{j_b}]^{(J)}{}_M\ ,\ (j_a, j_b = 1,\ 2) \tag{15.41}$$

by expressions such as

$$\boldsymbol{G}^{(J)}{}_M(j_a, j_b) = [\boldsymbol{b}^{+}_{\pi j_a} \times \boldsymbol{b}^{\sim}_{\pi j_b}]^{(J)}{}_M + [\boldsymbol{b}^{+}_{\nu j_a} \times \boldsymbol{b}^{\sim}_{\nu j_b}]^{(J)}{}_M\ ,\ (j_a, j_b = 1,\ 2) \tag{15.42}$$

one obtains by means of (15.40) just the sum $\boldsymbol{H_\pi} + \boldsymbol{H_\nu}$ plus a sum of terms such as (15.38). As an additional restriction in this special case one demands that these residual terms cancel out with the terms of $\boldsymbol{V_{\pi\nu}}$. This results in several conditions for the involved coefficients. Some authors leave a Majorana term (15.20) (Bonatsos, 1988, p. 78), (Frank and van Isacker, 1994, p. 369).

The remaining IBM2-Hamiltonian reads now analogously to (9.39) or (14.2)

$$\boldsymbol{H} = \varepsilon_n N + v_n N^2 + \tag{15.43}$$

$$(\varepsilon_d{}' + v_{nd} N)\boldsymbol{n}_{\boldsymbol{d},\, \pi+\nu} + v_d\, \boldsymbol{n}^2_{\boldsymbol{d},\, \pi+\nu} + v_r \boldsymbol{R}^2_{\pi+\nu} + v_t \boldsymbol{T}^2_{\pi+\nu} + v_j \boldsymbol{J}^2_{\pi+\nu} + v_q \boldsymbol{Q}^2_{\pi+\nu}$$

where in each operator the basis operators (15.41) are replaced by (15.42). These make up the same chains of algebras as in the IBM1. The formulation of these chains can be taken from (14.11), (14.17), (14.21) and table 14.1. They are put together in table 15.2 with usual markings.

Table 15.2. Chains of Lie algebras of the special cases I_1, II_1 and III_1 of the IBM2

Basis elements	Dimension n	Algebra	Special case
$\boldsymbol{G}^{(2)}_{\mu}(2,0)$, $\boldsymbol{G}^{(2)}_{\mu}(0,2)$, $\boldsymbol{G}^{(0)}_{0}(0,0)$	36	$u_{\pi+\nu}(6)$	
		$\cup$	
$\boldsymbol{G}^{(0)}_{0}(2,2)$, $\boldsymbol{G}^{(2)}_{\mu}(2,2)$, $\boldsymbol{G}^{(4)}_{\mu}(2,2)$	25	$u_{\pi+\nu}(5)$	
		$\cup$	I_1
$\boldsymbol{G}^{(3)}_{\mu}(2,2)$	10	$so_{\pi+\nu}(5)$	
		$\cup$	
$\boldsymbol{G}^{(1)}_{\mu}(2,2)$	3	$so_{\pi+\nu}(3)$	
$\boldsymbol{G}^{(2)}_{\mu}(2,0)$, $\boldsymbol{G}^{(2)}_{\mu}(0,2)$, $\boldsymbol{G}^{(0)}_{0}(0,0)$, $\boldsymbol{G}^{(0)}_{0}(2,2)$, $\boldsymbol{G}^{(3)}_{\mu}(2,2)$, $\boldsymbol{G}^{(4)}_{\mu}(2,2)$	36	$u_{\pi+\nu}(6)$	
		$\cup$	
$\boldsymbol{G}^{(2)}_{\mu}(2,0) + \boldsymbol{G}^{(2)}_{\mu}(0,2) - (\sqrt{7}/2)\, \boldsymbol{G}^{(2)}_{\mu}(2,2)$	8	$su_{\pi+\nu}(3)$	II_1
		$\cup$	
$\boldsymbol{G}^{(1)}_{\mu}(2,2)$	3	$so_{\pi+\nu}(3)$	
$\boldsymbol{G}^{(2)}_{\mu}(0,2)$, $\boldsymbol{G}^{(0)}_{0}(0,0)$, $\boldsymbol{G}^{(0)}_{0}(2,2)$, $\boldsymbol{G}^{(2)}_{\mu}(2,2)$, $\boldsymbol{G}^{(4)}_{\mu}(2,2)$	36	$u_{\pi+\nu}(6)$	
		$\cup$	
$\boldsymbol{G}^{(2)}_{\mu}(2,0) + \boldsymbol{G}^{(2)}_{\mu}(0,2)$	15	$so_{\pi+\nu}(6)$	
		$\cup$	III_1
$\boldsymbol{G}^{(3)}_{\mu}(2,2)$	10	$so_{\pi+\nu}(5)$	
		$\cup$	
$\boldsymbol{G}^{(1)}_{\mu}(2,2)$	3	$so_{\pi+\nu}(3)$	

As discussed in the preceding chapter every Lie algebra generates a set of basis functions. Their members are labelled with quantum numbers, which appear partly in the eigenvalues of the Casimir operators. In the IBM1 there is – except for *su*(3) – only one quantum number per algebra, which is relevant for the Casimir operator. In the IBM2 there are mostly two such numbers as the investigation of van Isacker et al. (1984) shows. For the special case I_1 (table 15.2) result the following quantum numbers

$$\left| \begin{array}{lllll} u_{\pi+\nu}(6) \supset u_{\pi+\nu}(5) \supset & so_{\pi+\nu}(5) & \supset & so_{\pi+\nu}(5) & \\ & & & & \\ [N_1, N_2] \quad [n_1, n_2] & [\tau_1, \tau_2], n_{\Delta 1}, n_{\Delta 2} & & [J] & \end{array} \right\rangle \quad \mathrm{I}_1 \qquad (15.44)$$

with $N_1 + N_2 = N$ $(N_1 \geq N_2)$, $n_1, + n_2 = n_{d\pi} + n_{d\nu}$.

The quantum numbers in square brackets characterise the eigenvalues of the Casimir operators (see 15.47) and the remaining numbers serve to the classification of the states. Analogously to the IBM1, the relations

$$n_1 \leq N_1,\ n_2 \leq N_2,\ \tau_1 \leq n_1,\ \tau_2 \leq n_2 \qquad (15.45)$$

hold together with further conditions for these quantum numbers and for the angular momentum. Iachello and Arima (1987, p. 152) have put them together.

By cancelling the coefficients v_r and v_q (15.43), one obtains analogously to the IBM1 (section 14.4) the combination of Casimir operators which belong to the first chain I_1 in table 15.2. In agreement with (14.37) the Hamilton operator can be written as

$$\boldsymbol{H}^{(\mathrm{I}_1)} =$$

$$\alpha_1\, \boldsymbol{C}_{\mathbf{1},\boldsymbol{u}\,\mathbf{(6)},\,\pi+\nu} + \alpha_2\, \boldsymbol{C}_{\mathbf{2},\boldsymbol{u}\,\mathbf{(6)}\,,\,\pi+\nu} + \beta\, \boldsymbol{C}_{\mathbf{1},\boldsymbol{u}\,\mathbf{(6)}\,,\,\pi+\nu}\, \boldsymbol{C}_{\mathbf{1},\boldsymbol{u}\,\mathbf{(5)}\,,\,\pi+\nu} + \qquad (15.46)$$

$$\gamma_1\, \boldsymbol{C}_{\mathbf{1},\boldsymbol{u}\,\mathbf{(5)}\,,\,\pi+\nu} + \gamma_2\, \boldsymbol{C}_{\mathbf{2},\boldsymbol{u}\,\mathbf{(5)}\,,\,\pi+\nu} + \delta\, \boldsymbol{C}_{\mathbf{2},\boldsymbol{so}\,\mathbf{(5)}\,,\,\pi+\nu} + \varepsilon\ \boldsymbol{C}_{\mathbf{2},\boldsymbol{so}\,\mathbf{(3)}\,,\,\pi+\nu}\,.$$

It contains Casimir operators which are formed by the operators $\boldsymbol{G}^{(J\ )}{}_{M}(j_a,\ j_b)$ (15.42) in the same way as the operators in table 14.2. The formulas for eigenvalues of Casimir operators depending on quantum numbers are given by Iachello and Arima (1987, p. 37, p. 158). According to (15.46) the following eigenenergies for the special case I_1 result

$$E_{\mathrm{I}_1} = \alpha_1\,(N_1 + N_2) + \alpha_2\,(N_1(N_1 + 5) + N_2\,(N_2 + 3)) +$$

$$\beta\,(N_1 + N_2\,)(n_1 + n_2) + \gamma_1(n_1 + n_2) + \qquad (15.47)$$

$$\gamma_2\,(n_1(n_1 + 4) + n_2\,(n_2 + 2)) + \delta\,(\tau_1\,(\tau_1 + 3) + \tau_2\,(\tau_2 + 1)) + \varepsilon\, J(J + 1).$$

For large boson numbers N, one obtains a multiplicity of possible splittings in N_1 and N_2 but only the lowest levels are important. Figure 15.12 shows a spectrum according to (15.47) for two active π-bosons and a ν-boson. The part with $N_1 = 3$ and $N_2 = 0$ corresponds to the IBM1 spectrum for the vibrational limit.

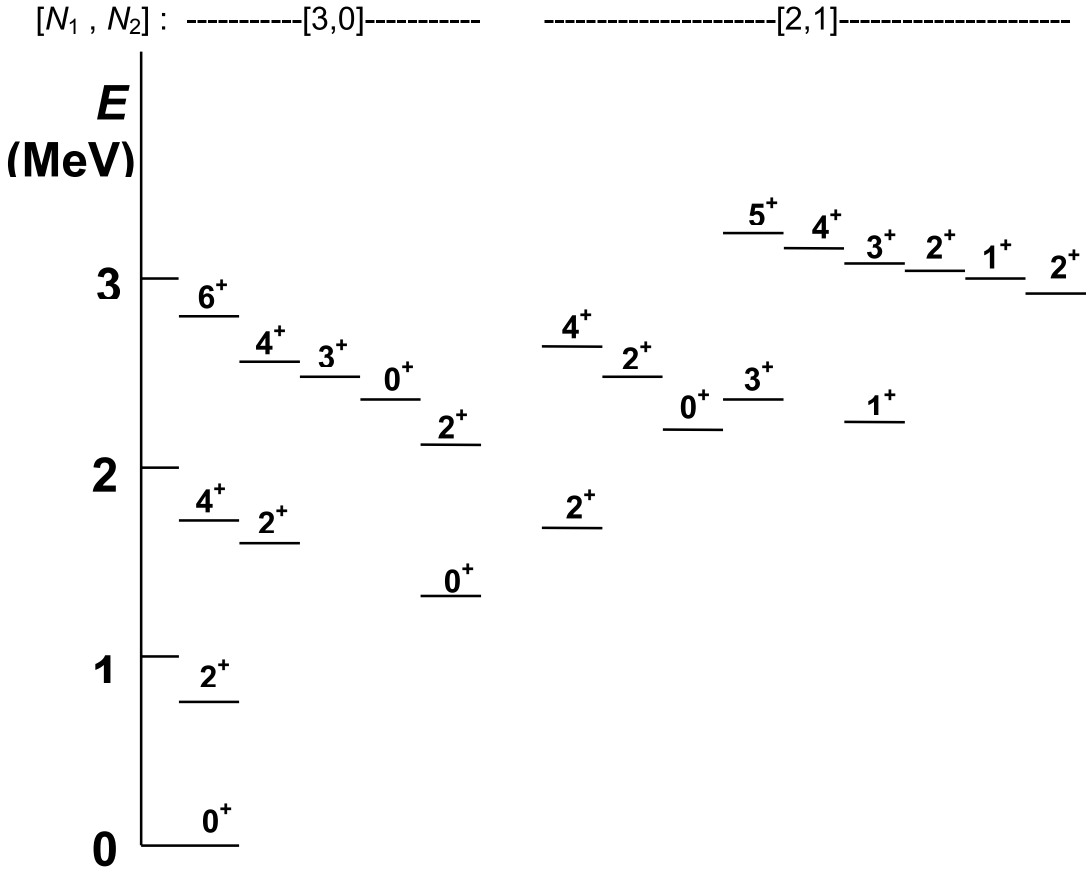

Figure 15.12. Typical spectrum for the I_1-special case of the IBM2. N_π = 2 and N_ν = 1 (or vice versa) have been employed (Iachello and Arima, 1987, p.159).

For the chains of the algebras II_1 and III_1 (table 15.2) there exist representations of eigenvalues corresponding to (15.46) and (15.47) (Iachello and Arima, 1987, p. 159). In appropriate publications further chains of algebras are mentioned, in which the higher algebras are direct sums of π- and ν -algebras.

16 The interacting boson-fermion model IBFM

In the first section the extension of the IBM will be outlined which admits odd numbers of protons and/or neutrons. Creation and annihilation operators for nucleons are introduced and the corresponding Hamiltonian is brought in the usual form. The second section deals with a version of the IBMF which employs the boson-boson interaction of the *u*(5)-limit. The simple case of one active *d*-boson coupled to a nucleon is discussed. In the third section we will explaine how the number of parameters in the boson-nucleon interaction can be reduced realistically. Comparisons of IBFM calculations with measurements will follow.

16.1 The Hamiltonian of the IBFM

In the preceding chapters the nuclear model was restricted to nuclei with even proton- and neutron numbers. However, most of the nuclei are composed of odd-even (oe) or odd-odd (oo) nucleon numbers.

In order to include these nuclei, the interacting boson model is extended by attaching a nucleon to the boson configuration. For oo-nuclei, one has to add a proton and a neutron. The number of bosons is fixed in the same way as in chapter 2. If the active bosons are holes, the nucleon has to be taken in the hole state as well.

Nucleons are known to have total spins j = 1/2, 3/2, 5/2, ... with regard to the nucleus and they can show positive or negative parity. Coupling a spherical basis state for bosons (chapter 3, (6.18)) to a nucleon state creates the simplest basis state for oe-nuclei like this

$$|N_B\, n_d\, \tau\, n_\Delta\, J_B\, j\, J\, M\rangle \equiv |[(N_B\, n_d\, \tau\, n_\Delta)^{(J_B)} \times j\,]^{(J\,)}{}_M\rangle. \tag{16.1}$$

For nucleons, there exist also creation and annihilation operators as follows

$$\boldsymbol{a}^{+}_{jm}\;,\; \boldsymbol{a}_{jm} \tag{16.2}$$

which are assigned to the nucleon state (j,m). Analogously to (6.19) the annihilation operator can be put in tensor form like this

$$\boldsymbol{a}^{\sim}{}_{jm} = (-1)^{j+m}\, \boldsymbol{a}_{j,-m}\;. \tag{16.3}$$

Because the nucleon operators (16.2) create (or annihilate) antisymmetric states, the commutation rules (5.34) and (5.35) are not valid and the commutators have to be replaced by anticommutators. In analogy with the IBM2 (section 15.1) one demands that $\boldsymbol{a}^{+}_{jm}$ and $\boldsymbol{a}^{\sim}{}_{jm}$ commute with the boson creation and annihilation operators. That is to say, bosons and nucleons are regarded as essentially different objects and one forgets that a boson represents a nucleon pair.

The Hamilton operator for bosons and nucleons contains, besides the interaction between the bosons, also nucleon-nucleon and nucleon-boson interactions. Analogously to (15.4) it reads

$$\boldsymbol{H} = \boldsymbol{H_B} + \boldsymbol{H_F} + \boldsymbol{V_{BF}}\,. \tag{16.4}$$

For the boson part $\boldsymbol{H_B}$, mostly the IBM1-Hamilton operator (7.17) is employed. The pure nucleon Hamiltonian $\boldsymbol{H_F}$ is constructed analogously to (7.3) and (7.6) as follows

$$\boldsymbol{H_F} = E_0 + \sum_j \varepsilon_j \sqrt{(2j + 1)}\, [\boldsymbol{a}^{+}_{j} \times \boldsymbol{a}^{\sim}_{j}]^{(0)} +$$

$$\sum_{J j j' j'' j'''} c^{(J)}_{J j j' j'' j'''} [[\boldsymbol{a}^{+}_{j} \times \boldsymbol{a}^{+}_{j'}]^{(J)} \times [\boldsymbol{a}^{\sim}_{j''} \times \boldsymbol{a}^{\sim}_{j'''}]^{(J)}]^{(0)}. \tag{16.5}$$

Henceforth we will restrict ourselves to oe-nuclei. Therefore, the last sum in (16.5) disappears because it does not contribute when it acts on a single nucleon. The quantity ε_j is the single-nucleon energy of the nucleon state (jm). In analogy with (15.7) the boson-fermion interaction can be written as

$$\boldsymbol{V_{BF}} = \sum_{l_1 l_2 j_1 j_2 j} v^{(j)}_{l_1 j_1 l_2 j_2} [[\boldsymbol{b}^{+}_{l_1} \times \boldsymbol{a}^{+}_{j_1}]^{(j)} \times [\boldsymbol{b}^{\sim}_{l_2} \times \boldsymbol{a}^{\sim}_{j_2}]^{(j)}]^{(0)} \tag{16.6}$$

with (l_1, l_2) = 0, 2 and (j_1, j_2) = 1/2, 3/2, 5/2, ... The b-operators concern bosons. The coefficients $v^{(j)}_{l_1 j_1 l_2 j_2}$ are matrix elements of the boson-fermion interaction this way

$$v^{(j)}_{l_1 j_1 l_2 j_2} = \langle b_{l_1}\, a_{j_1}\, j \mid \boldsymbol{v_{BF}} \mid b_{l_2}\, a_{j_2}\, j\rangle. \tag{16.7}$$

Analogously to (15.9) there is linear dependency between some of the quantities $v^{(j)}_{l_1 j_1 l_2 j_2}$. Frequently it is sufficient to employ only one nucleon spin $j_1 = j_2 = j_n$. The expression (16.6) can be recoupled by means of (A4.1), which yields

$$\boldsymbol{V_{BF}} = \sum_{l_1 l_2 j_1 j_2 J} c^{(J)}_{l_1 j_1 l_2 j_2} [[\boldsymbol{b}^{+}_{l_1} \times \boldsymbol{b}^{\sim}_{l_2}]^{(J)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}^{\sim}_{j_2}]^{(J)}]^{(0)}. \tag{16.8}$$

The coefficients $c^{(J)}_{l_1 j_1 l_2 j_2}$ are linear combinations of $v^{(j)}_{l_1 j_1 l_2 j_2}$ and their factors contain 9-j symbols. We write (16.8) in detail employing d- and s-operators

$$\boldsymbol{V_{BF}} = \sum_j c^{(0)}_{00jj} [[\boldsymbol{s}^{+} \times \boldsymbol{s}]^{(0)} \times [\boldsymbol{a}^{+}_{j} \times \boldsymbol{a}^{\sim}_{j}]^{(0)}]^{(0)} +$$

$$\sum_j c^{(0)}_{22jj} [[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(0)} \times [\boldsymbol{a}^{+}_{j} \times \boldsymbol{a}^{\sim}_{j}]^{(0)}]^{(0)} + \sum_{j_1 j_2} c^{(2)}_{02 j_1 j_2} \cdot \tag{16.9}$$

$$[([\boldsymbol{s}^{+} \times \boldsymbol{d}^{\sim}]^{(2)} + [\boldsymbol{d}^{+} \times \boldsymbol{s}]^{(2)} + (c^{(2)}_{22 j_1 j_2} / c^{(2)}_{02 j_1 j_2})\, [\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(2)}) \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}^{\sim}_{j_2}]^{(2)}]^{(0)} +$$

$$\sum_{j_1 j_2,\, J = 1,3,4} c^{(J)}_{22 j_1 j_2} [[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(J)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}^{\sim}_{j_2}]^{(J)}]^{(0)}.$$

The expressions $[[\boldsymbol{d}^{+} \times \boldsymbol{d}^{\sim}]^{(J)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}^{\sim}_{j_2}]^{(J)}]^{(0)}$ (16.10)

in the last term of (16.9) are frequently written in another form. According to (A1.4) we have

$$[\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}^{\sim}_{j_2}]^{(J)}_{M} = (-1)^{j_1 + j_2 - J} \sum_{m_1 m_2} (j_2\, m_2\, j_1\, m_1 \mid J\, M)\, \boldsymbol{a}^{+}_{j_1}\, \boldsymbol{a}^{\sim}_{j_2}\,. \tag{16.11}$$

Now we are defining the operator of normal ordering of nucleon operators as follows

$$: \boldsymbol{a}\tilde{}_{j_2}\, \boldsymbol{a}^{+}_{j_1} : \equiv \boldsymbol{a}^{+}_{j_1}\, \boldsymbol{a}\tilde{}_{j_2}. \qquad (16.12)$$

Therefore, we can write

$$[\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}\tilde{}_{j_2}]^{(J)}{}_{M} = (-1)^{j_1 + j_2 - J} :[\boldsymbol{a}\tilde{}_{j_2} \times \boldsymbol{a}^{+}_{j_1}]^{(J)}{}_{M} :. \qquad (16.13)$$

This means that the right hand side of (16.13) first is represented formally by means of (A1.1) and then the interchanging (16.12) is performed. Now the expression (16.10) is written with the aid of (A4.1) as follows

$$[[\boldsymbol{d}^{+} \times \boldsymbol{d}\tilde{}\,]^{(J)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}\tilde{}_{j_2}]^{(J)}]^{(0)} = \qquad (16.14)$$

$$\sum_{j} (2J + 1)(2j + 1) \begin{Bmatrix} 2 & 2 & J \\ j_1 & j_2 & J \\ j & j & 0 \end{Bmatrix} (-1)^{-j_2 - j} :[[\boldsymbol{d}^{+} \times \boldsymbol{a}\tilde{}_{j_2}]^{(j)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{d}\tilde{}\,]^{(j)}]^{(0)}: .$$

Therefore, the last sum in (16.9) obtains the following form

$$\sum_{j_1 j_2,\, J = 1,3,4} c^{(J)}{}_{22 j_1 j_2} [[\boldsymbol{d}^{+} \times \boldsymbol{d}\tilde{}\,]^{(J)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}\tilde{}_{j_2}]^{(J)}]^{(0)} =$$

$$\sum_{j\, j_1 j_2} \Lambda^{(j)}{}_{j_2 j_1} :[[\boldsymbol{d}^{+} \times \boldsymbol{a}\tilde{}_{j_2}]^{(j)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{d}\tilde{}\,]^{(j)}]^{(0)} : . \qquad (16.15)$$

With the help of the number operators (6.35) up to (6.37) such as

$$[\boldsymbol{s}^{+} \times \boldsymbol{s}]^{(0)} = \boldsymbol{N} - \boldsymbol{n_d}, \quad [\boldsymbol{d}^{+} \times \boldsymbol{d}\tilde{}\,]^{(0)} = \boldsymbol{n_d}/\sqrt{5}, \quad [\boldsymbol{a}^{+}_{j} \times \boldsymbol{a}\tilde{}_{j}]^{(0)} = n_j/\sqrt{(2j + 1)}, \quad (16.16)$$

with the usual markings (Iachello and van Isacker, 1991, p. 12) like this

$$c^{(0)}{}_{00jj}\sqrt{5} - c^{(0)}{}_{22jj} = A_j, \quad c^{(2)}{}_{02 j_1 j_2} = \Gamma_{j_1 j_2}, \quad (c^{(2)}{}_{22 j_1 j_2} / c^{(2)}{}_{02 j_1 j_2}) = \chi_{j_1 j_2} \qquad (16.17)$$

and with the quadrupole operator (15.17)

$$\boldsymbol{Q}_B(\chi_{j_1 j_2})_\mu = [\boldsymbol{d}^{+} \times \boldsymbol{s}]^{(2)}{}_\mu + [\boldsymbol{s}^{+} \times \boldsymbol{d}\tilde{}\,]^{(2)}{}_\mu + \chi_{j_1 j_2} [\boldsymbol{d}^{+} \times \boldsymbol{d}\tilde{}\,]^{(2)}{}_\mu \qquad (16.18)$$

one obtains for $\boldsymbol{V_{BF}}$ (16.9) the following expression

$$\boldsymbol{V_{BF}} = \sum_j c^{(0)}{}_{00jj}\, \boldsymbol{N}\, \boldsymbol{n_j} - \sum_j A_j\, \boldsymbol{n_d}\, \boldsymbol{n_j}/\sqrt{(5(2j + 1))} +$$

$$\sum_{j_1 j_2} \Gamma_{j_1 j_2} [\boldsymbol{Q}_B(\chi_{j_1 j_2}) \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{a}\tilde{}_{j_2}]^{(2)}]]^{(0)} + \qquad (16.19)$$

$$\sum_{j\, j_1 j_2} \Lambda^{(j)}{}_{j_2 j_1} :[[\boldsymbol{d}^{+} \times \boldsymbol{a}\tilde{}_{j_2}]^{(j)} \times [\boldsymbol{a}^{+}_{j_1} \times \boldsymbol{d}\tilde{}\,]^{(j)}]^{(0)} : .$$

The second term on the right hand side of (16.19) is named monopole term and the following are the quadrupole term and the exchange term.

16.2 The $u(5)$-limit of the IBFM

If the part $\boldsymbol{H_B}$ of the Hamiltonian (16.4) has the special form (14.31) of the $u(5)$-limit, the state (16.1) is an eigenfunction of $\boldsymbol{H_B}$ with the following eigenvalue (10.17)

$$E^{(I)}{}_B = \varepsilon_n N + v_n N^2 + (\varepsilon_d' + v_{nd} N) n_d + v_d n_d^2 + v_t\, \tau(\tau + 3) + v_j J_B(J_B + 1). \quad (16.20)$$

It can be written in the form (14.38) as well. The matrix of the operator $\boldsymbol{H_B}$ on the basis of functions (16.1) has therefore only diagonal elements. Naturally $\boldsymbol{H_F}$ (for one single nucleon) is diagonal and has the eigenvalues ε_j.. In order to find the eigenenergies of the whole operator $\boldsymbol{H}$ (16.4) essentially only $\boldsymbol{V_{BF}}$ has to be diagonalised (see section 12.1). We investigate the influence of the individual terms of $\boldsymbol{V_{BF}}$ (16.19). Because the monopole term on the right hand side of (16.19) contains only number operators it gives an identical contribution to every diagonal matrix element with the same number of d-bosons and the same nucleon state $|j\, m\rangle$. The quadrupole term in $\boldsymbol{V_{BF}}$ contains the operators

$$\sum_{j_1 j_2} \Gamma_{j_1 j_2} [[\boldsymbol{d}^+ \times \boldsymbol{s} + \boldsymbol{s}^+ \times \boldsymbol{d}^\sim]^{(2)} \times [\boldsymbol{a}^+_{j_1} \times \boldsymbol{a}^\sim_{j_2}]^{(2)}]^{(0)} \text{ and} \quad (16.21)$$

$$\sum_{j_1 j_2} \chi_{j_1 j_2} \Gamma_{j_1 j_2} [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)} \times [\boldsymbol{a}^+_{j_1} \times \boldsymbol{a}^\sim_{j_2}]^{(2)}]^{(0)}. \quad (16.22)$$

The expression (16.21) generates non-diagonal elements in the energy matrix. Experience shows that its contributions can be neglected if ε'_d (16.20) is relatively large. The term (16.22) contributes to the diagonal matrix elements, which we now look into for the situation of a single d-boson coupled to a nucleon state $|\, j\, m\rangle$. Such a diagonal term reads

$$\langle\, s^{N-1}\, [d \times j]^{(J)}{}_M \,|\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(2)} \times [\boldsymbol{a}^+_j \times \boldsymbol{a}^\sim_j]^{(2)}]^{(0)} \,|\, s^{N-1}\, [d \times j]^{(J)}{}_M \rangle = \quad (16.23)$$

$$\sum_{J'} 5(2J' + 1) \begin{Bmatrix} 2 & 2 & 2 \\ j & j & 0 \\ J' & J' & 0 \end{Bmatrix} \langle\, [d \times j\,]^{(J)}{}_M |\, [[\boldsymbol{d}^+ \times \boldsymbol{a}^+_j\,]^{(J')} \times [\boldsymbol{d}^\sim \times \boldsymbol{a}^\sim_j]^{(J')}]^{(0)} |[d \times j\,]^{(J)}{}_M \rangle =$$

$$\sum_{J'M'} \sqrt{5}\, (-1)^{j+2J'-M'} \begin{Bmatrix} 2 & j & J' \\ j & 2 & 2 \end{Bmatrix} \langle\, [d \times j\,]^{(J)}{}_M \,|\, [\boldsymbol{d}^+ \times \boldsymbol{a}^+_j\,]^{(J')}{}_{M'}\, [\boldsymbol{d}^\sim \times \boldsymbol{a}^\sim_j]^{(J')}{}_{-M'} |\, [d \times j\,]^{(J)}{}_M \rangle.$$

The first transformation in (16.23) is a recoupling according to (A4.1) and the second employs the relations (4.2), (A1.1) and (A1.16). Furthermore, because of (16.3), (A1.1), (A1.5) and (A1.10) the following equation holds

$$[\boldsymbol{d}^\sim \times \boldsymbol{a}^\sim_j]^{(J')}{}_{-M'} |\, [d \times j\,]^{(J)}{}_M \rangle =$$

$$\sum_{\mu' m' \mu m} (-1)^{\mu'+j+m'}\, (2\, \mu'\, j\, m' \,|\, J', -M')(2\, \mu\, j\, m \,|\, J\, M) \boldsymbol{d_{-\mu'}} \boldsymbol{a_{j\,-m'}} |\, d_\mu\, a_{jm} \rangle =$$

$$\sum_{\mu' m' \mu m} (-1)^{j-M'}\, (2, -\mu\, j, -m \,|\, J', -M')(2\, \mu\, j\, m \,|\, J\, M)\, \delta_{\mu, -\mu'}\, \delta_{m, -m'} =$$

$$(-1)^{2j-J'-M'}\, \delta_{J'J}\, \delta_{M'M}\,. \quad (16.24)$$

Analogously we have

$$\langle [d \times j\,]^{(J)}{}_M \,|\, [\boldsymbol{d}^+ \times \boldsymbol{a}^+_j]^{(J')}{}_{-M'} = \delta_{J'J}\, \delta_{M'M}\,. \quad (16.25)$$

Therefore, the term (16.22) yields the following contribution to the diagonal matrix element

$$\Delta E_{n_d=1} = -\chi_{jj}\,\Gamma_{jj}\sqrt{5}\,(-1)^{J-j}\left\{\begin{matrix}2 & j & J\\ j & 2 & 2\end{matrix}\right\}. \qquad (16.26)$$

The energy of the state represented by a j-nucleon coupled up to a d-boson splits up into $2j + 1$ or 5 (if $j > 2$) levels starting from the energy ($\varepsilon_d' + v_{nd}\,N$) + $v_d + 4v_t + 6v_j$ (16.20). The splitting is proportional to the geometrical factors (16.26), which depend on J ($2 + j \geq J \geq |2 - j|$). The exchange term in $\boldsymbol{V_{BF}}$ (16.19) yields a similar multiplicity of terms as (16.26).

Figure 16.1 shows a comparison between measured and calculated energies of the nucleus ${}^{133}_{57}La_{76}$ for d-boson configurations of the $u(5)$-limit. The 2^+-state of ${}^{152}_{56}Ba_{76}$ (n_d = 1) is split up into 5 levels by coupling a j = 11/2-proton according to (16.26).

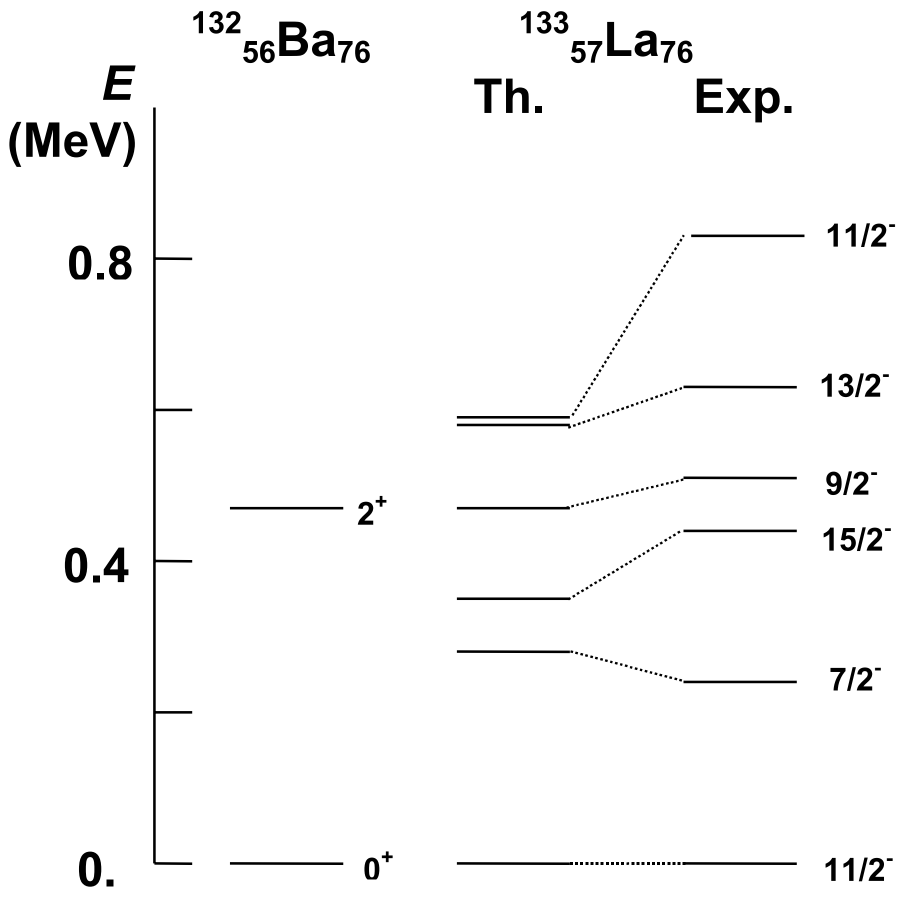

Figure 16.1. Comparison between measured and calculated energies of low-lying levels of ${}^{133}La$. The energies of the 5 excited levels are calculated starting from the 2^+-level of ${}^{132}Ba$ (consisting of one d-boson in the $u(5)$-limit) with the aid of equation (16.26) and with the coefficient $\chi_{jj}\,\Gamma_{jj}$ = -0.94 MeV (Scholten, 1980, p. 98).

Figure 16.2 compares experimental data with results of IBFM calculations in the $u(5)$-limit of two Eu-isopotes.

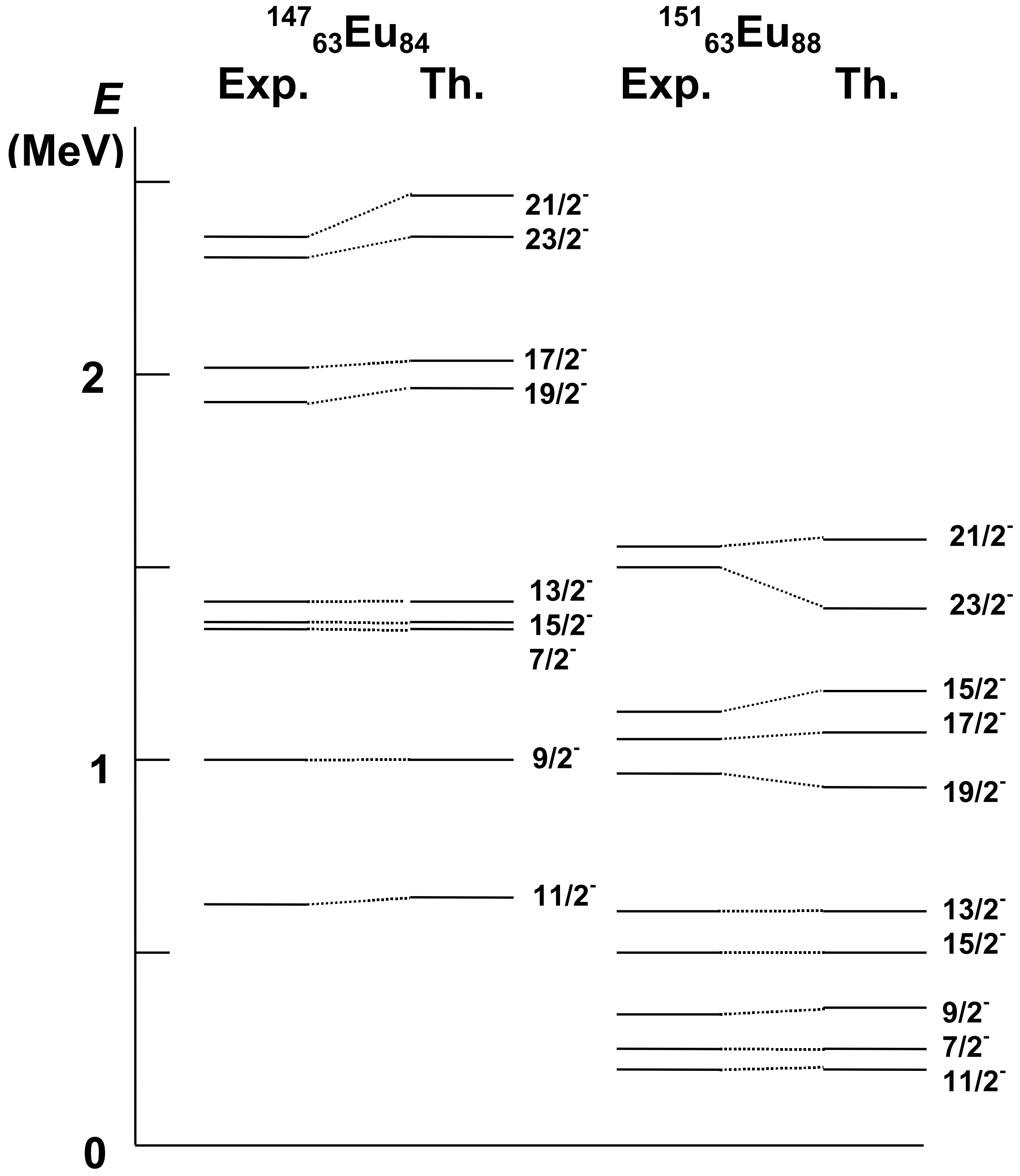

Figure 16.2. Comparison between measured and calculated energy levels of ^{147}Eu and ^{151}Eu (Casten, 1980, p. 199). The calculations are performed according to the equation (16.19) and Arima and Iachello (1976b) respectively, by coupling a $11/2^-$-proton up to $u(5)$-states and by fitting the parameters. The ground states ($J = 11/2^-$) are drawn in, taking into account the binding energies.

Nuclei with boson configurations belonging to the other two special cases of dynamical symmetries are treated analogously.

16.3 Numerical treatment of the IBFM. Comparison with experimental data

The general operators $\boldsymbol{H_B}$ and $\boldsymbol{H_F}$ (16.4) and $\boldsymbol{V_{BF}}$ (16.19) comprise so many free parameters that the model becomes cumbersome. One has to reduce them suitably similarly as in the IBM2. In this section we deal specially with the term $\boldsymbol{V_{BF}}$.

Owing to the microscopic theory, which connects the IBM with the shell model, and making use of the generalised seniority principle, Scholten (1981, p.286) gives the following instruction for the coefficients in (16.19),, which represents an approximation

$$A_j = -\sqrt{(5(2j + 1))}\, A_0,$$

$$\Gamma_{j_1 j_2} = \sqrt{5}\gamma_{j_1 j_2}\, \Gamma_0, \quad \text{with } \gamma_{j_1 j_2} = (u_{j_1} u_{j_2} - v_{j_1} v_{j_2})Q_{j_1 j_2}, \qquad (16.27)$$

$$\Lambda^{(j)}{}_{j_1 j_2} = -2\sqrt{(5/(2j + 1))}\, \beta_{j_1 j}\, \beta_{j_2 j}\, \Lambda_0 \quad \text{with } \beta_{j'j} = (u_{j'} v_j + v_{j'} u_j)Q_{j'j}.$$

The quantities $Q_{j'j}$ in (16.27) are reduced matrix elements of the quadrupole operator in the single-nucleon basis like this

$$Q_{j'j} = \langle j' \,||\, Y^{(2)} \,||\, j \rangle.$$

The quantities v_j^2 are occupation probabilities for the nucleon states (Ring and Schuck, 1980), (Otsuka and Arima, 1978). The relation $u_j = \sqrt{(1 - v_j^2)}$ holds. Apart from the constant term, the parameters of $\boldsymbol{V_{BF}}$ (16.19) are reduced to A_0, Γ_0 and Λ_0 according to (16.27).

One finds the parameters of $\boldsymbol{H_B}$ in practice by fitting IBM1 calculations to the next lighter (or heavier) even-even nucleus which contains one nucleon (or nucleon hole) less than the even-odd nucleus.

Similarly as in chapter 12 the matrix elements of the Hamilton operator can be formed on the basis (16.1). By diagonalising, one obtains the eigenenergies and eigenstates.

Figure 16.3 shows results of such an IBFM calculation for ${}^{101}_{45}Rh_{56}$ compared with the measured level energies. States with positive parity are treated as a coupling of an IBM1 state to a $g_{9/2}$-proton.

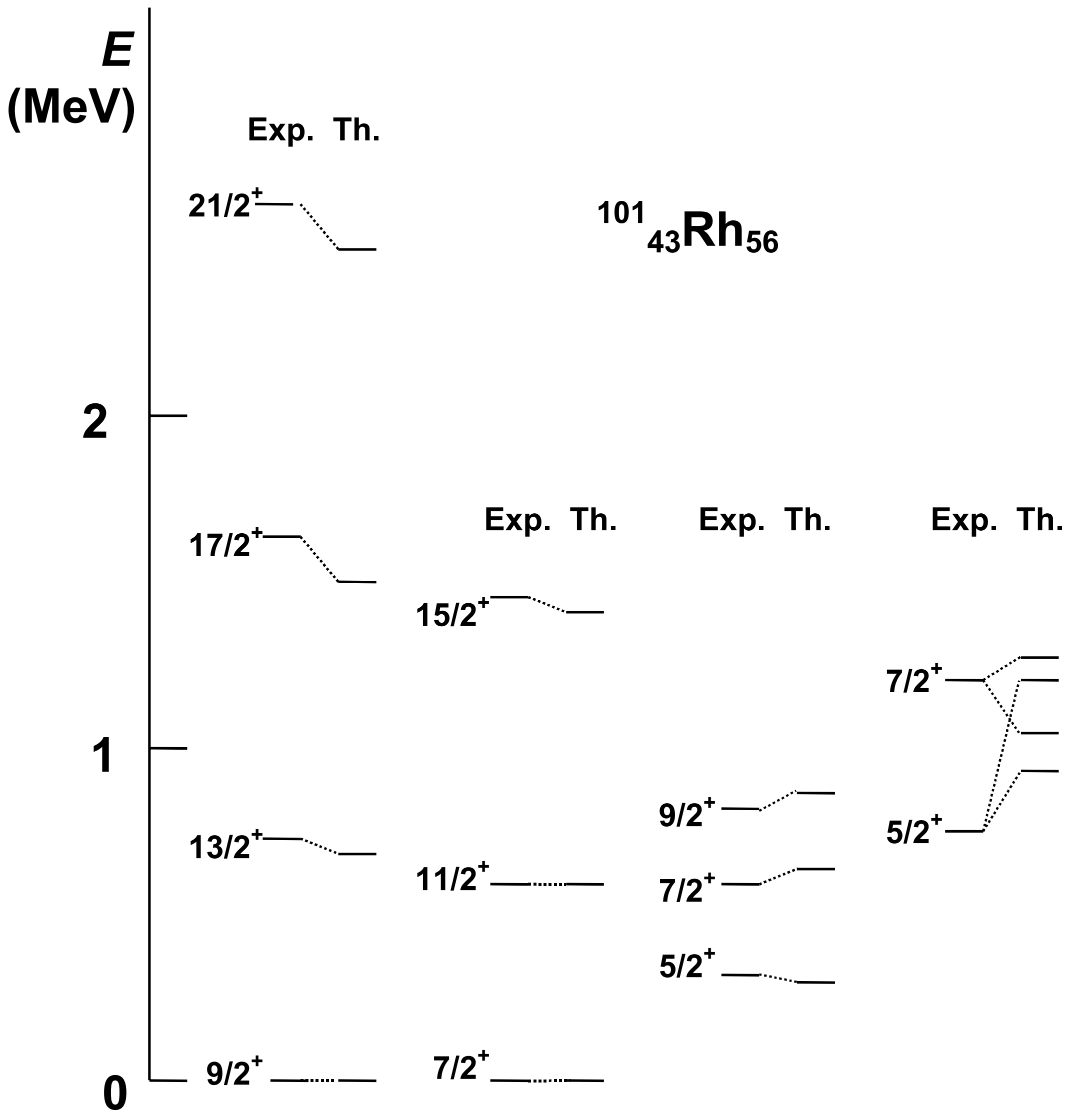

Figure 16.3. Comparison between measured and calculated low-lying levels with positive parity of ^{101}Rh (Scholten, 1981, p. 289). Not all experimental spin assignments are unambiguous. In the calculation a $g_{9/2}$-proton is coupled to a $^{100}_{44}Ru_{56}$ nucleus which has been treated with the IBM1 making use of the parameters ε = 0.733 MeV, c_0' = - 0.415 MeV, c_2' = - 0.283 MeV, c_4' = - 0.024 MeV, v_2 = 0.175 MeV and v_0 = - 0.31 MeV (7.31). For the fermion-boson term $\boldsymbol{V_{BF}}$ (16.19 and 16.27) the value v_j^2 = 0.5 has been chosen for which reason the quadrupole term vanishes.

The low-lying states of ^{191}Ir are calculated using a Hamiltonian containing an IBM1 part for bosons and parts with the proton state $(h_{11/2}\text{-})^{-1}$. In figure 16.4, the results for negative parity are compared with experimental data.

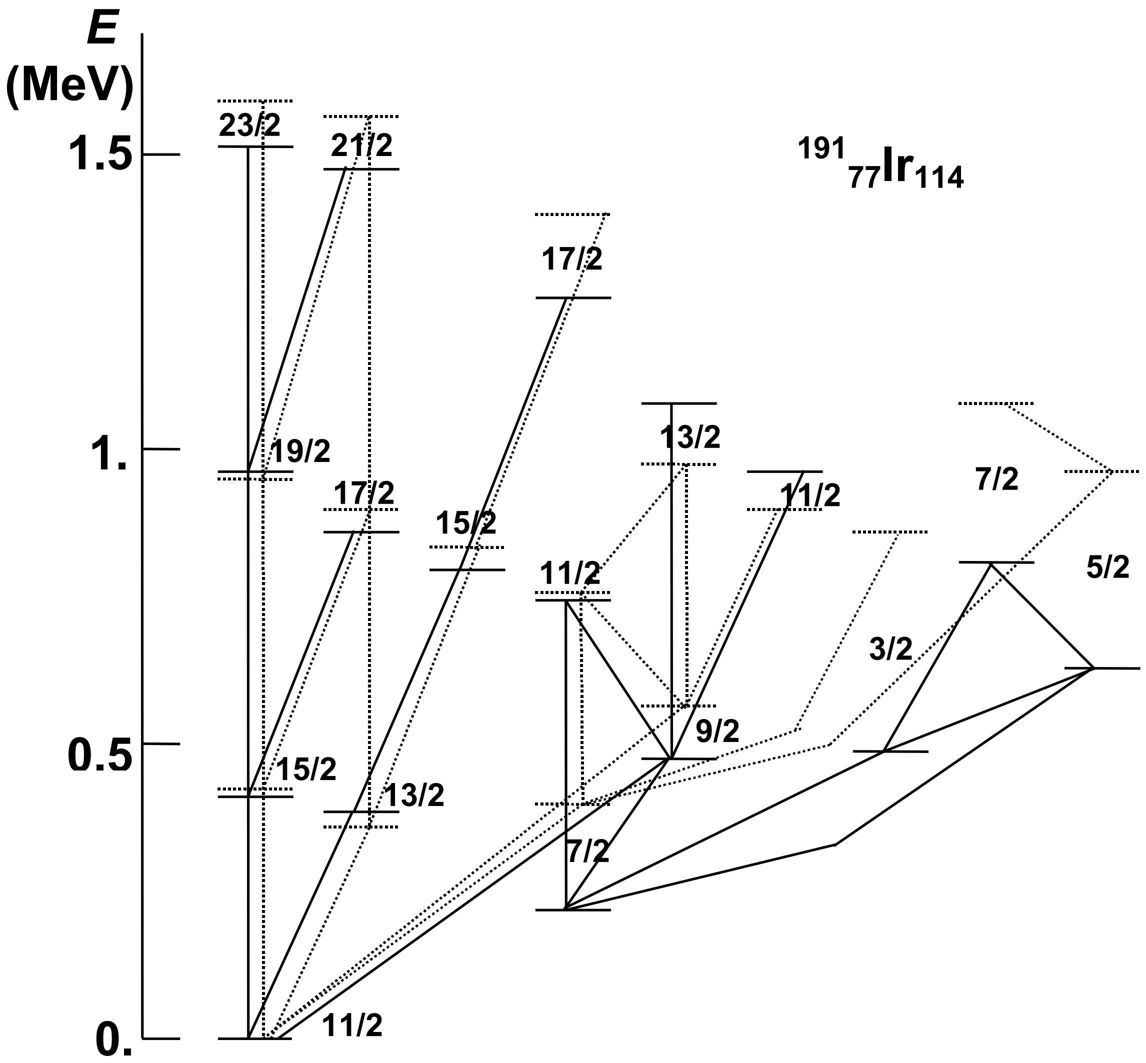

Figure 16.4. Comparison between calculated and experimental spectra of the negative-parity states in ^{191}Ir (Bijker and Dieperink, 1982, p. 229). The broken lines denote calculated data. The levels are interpreted in the IBFM model as a $h_{11/2}$- proton hole state coupled to the $^{192}_{78}Pt_{114}$ core. The lines connecting the levels represent large E2 transitions by radiation.

Isotopes of gold of the odd-even type with 106 up to 116 neutrons have been treated as a coupling of an even-even Pt-core to a $h_{9/2}^{-}$-proton. Figure 16.5 shows the lowest levels compared with experimental data.

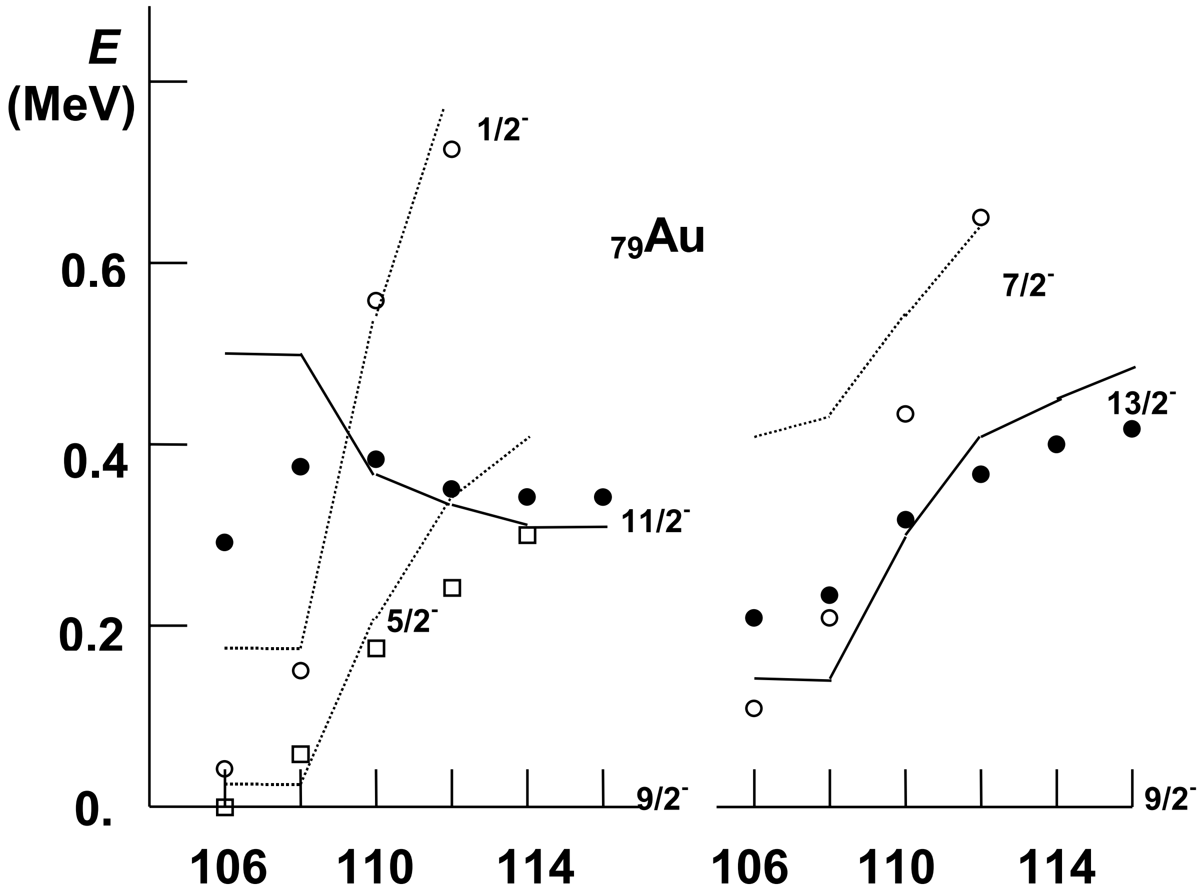

Figure 16.5. Comparison between measured and calculated spectra with negative parity in $_{79}$ Au-isotopes (Bijker and Dieperink, 1982, p. 231). The neutron numbers comprise 104 up to 116. For the calculation, a $h_{9/2}$-proton has been coupled to the corresponding $_{78}$Pt-cores.

Figure 16.6 represents an IBFM-calculation for numerous levels with positive parity of ^{97}Tc. The even-even nucleus is coupled to a proton with two different shell states.

The IBFM comprises more dynamic symmetries than the IBM2. Because this book is only an introduction to the interacting boson model, we leave them out of consideration.

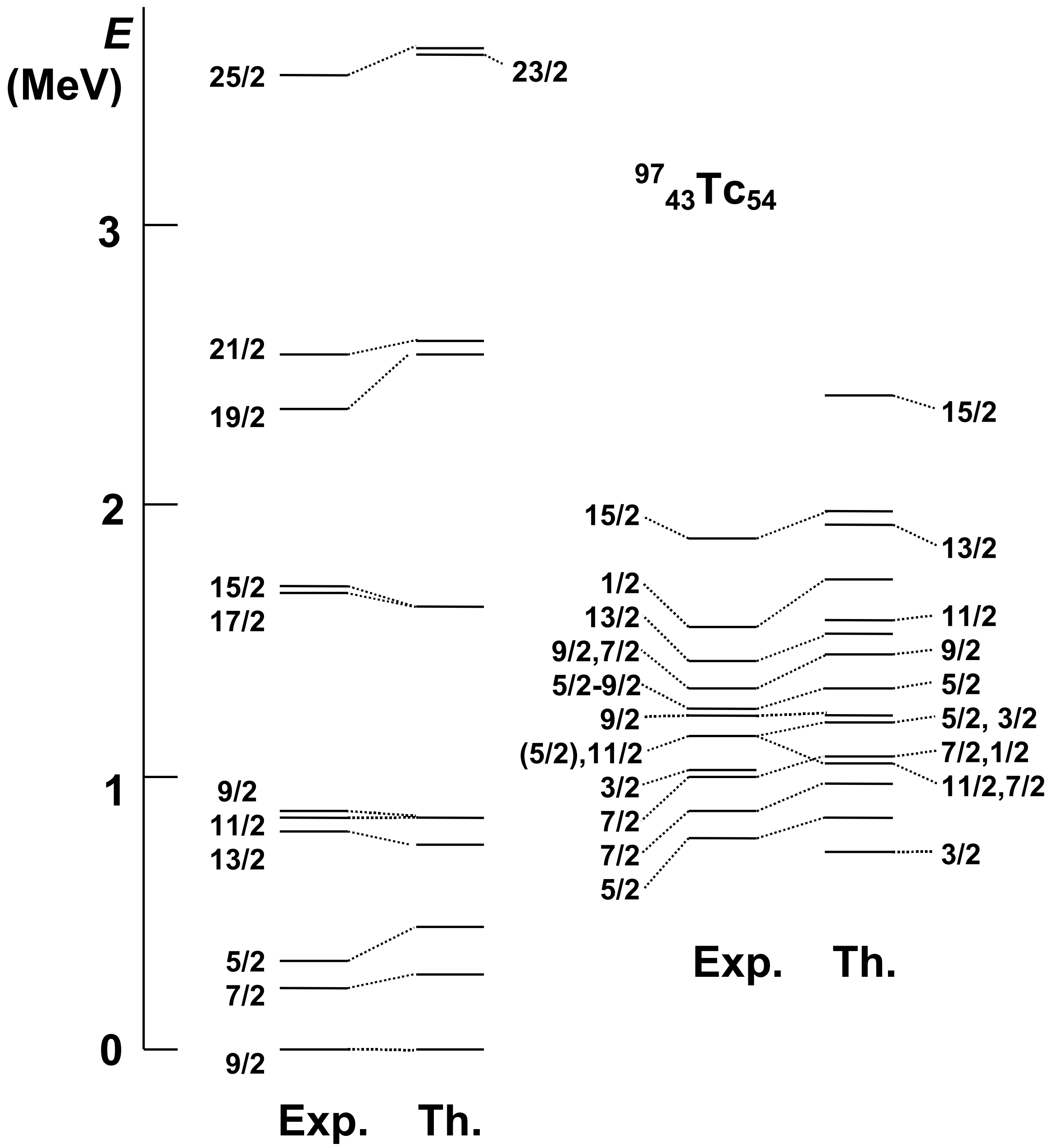

Figure 16.6. Comparison between the measured and the calculated spectrum with positive parity of ^{97}Tc (von Brentano, 1981, p. 307). In the calculation, a proton with the states $g_{9/2}$ and $d_{5/2}$ has been coupled to ^{96}Mo. This nucleus has been treated with the IBM1 by means of the parameters ε = 0.795 MeV, c_0' = - 0.25 MeV, c_2' = - 0.085 MeV, c_4' =0.047 MeV, v_2 = 0.113 MeV and v_0 = 0 MeV (7.31). The fermion-boson term $\boldsymbol{V_{BF}}$ (16.18) has been calculated with expressions, which correspond largely to the equations (16.19) and (16.27).

Appendix

A1 Clebsch-Gordan coefficients and 3-*j* symbols

Some properties of the Clebsch-Gordan coefficients will be reviewed which are employed in the treatment of the IBM on hand. Descriptions of this topic in detail are given by Condon and Shortley (1953) and Talmi (1993).

Clebsch-Gordan coefficients arise when state functions containing a definite spin or angular momentum are coupled and constitute a joint state with good spin quantum numbers. Let the normalised and orthogonal function φ_{jm} describe the state of a particle or boson with spin or angular momentum j and projection m. In the language of quantum mechanics this means that φ_{jm} is an eigenfunction of the operators $\boldsymbol{J}^2$ and $\boldsymbol{J_z}$.

Staring from the functions $\varphi_{j_am_a}$ and $\varphi_{j_bm_b}$ which belong to different objects a joint state $\Phi_{JM}(j_a\ ,\, j_b)$ can be made up which is also an eigenfunction of the operators $\boldsymbol{J}^2$ and $\boldsymbol{J_z}$. Its total spin is characterized by J with the projection M. The new state reads

$$\Phi_{JM}(j_a\ ,\, j_b) = \sum_{m_a\, m_b} (j_a\ m_a\ j_b\ m_b \mid J\ M)\ \varphi_{j_am_a}\varphi_{j_bm_b} \qquad \text{(A1.1)}$$

with $\quad m_a + m_b = M\ $ and $\mid j_a - j_b \mid\ \leq J \leq j_a + j_b$. (A1.2)

The factors $(j_a\ m_a\ j_b\ m_b \mid J\ M)$ are named Clebsch-Gordan coefficients or vector coupling coefficients. They are real and vanish if the conditions (A1.2) are violated. The coupling (A1.1) is frequently written as

$$[\varphi_{j_am_a} \times \varphi_{j_bm_b}]^{(J)}{}_M\ \equiv \Phi_{JM}(j_a\ ,\, j_b). \qquad \text{(A1.3)}$$

Clebsch-Gordan coefficients meet the following symmetry conditions

$$(j_a\ m_a\ j_b\ m_b \mid J\ M)\ = (-1)^{j_a+ j_b- J}\,(j_b\ m_b\ j_a\ m_a \mid J\ M), \qquad \text{(A1.4)}$$

$$(j_a\ m_a\ j_b\ m_b \mid J\ M) = (-1)^{j_a+ j_b - J}(j_a,\ -m_a,\ j_b,\ -m_b \mid J,\ -M), \qquad \text{(A1.5)}$$

$$(j_a\ m_a\ j_b\ m_b \mid J\ M) = (-1)^{j_a- m_a}\,((2J + 1)/(2j_b + 1))^{1/2}(j_a\ m_a\ J,\ -M \mid j_b,\ -m_b). \qquad \text{(A1.6)}$$

For $j_a = j_b = j$ specially we can write

$$(j\ m\ j\ m' \mid J\ M) = (-1)^{2j - J}\,(j\ m'\ j\ m \mid J\ M). \qquad \text{(A1.7)}$$

If additionally $m = m'$ holds we have

$$(j\ m\ j\ m \mid J\ M) = (-1)^{2j - J}\,(j\ m\ j\ m \mid J\ M). \qquad \text{(A1.8)}$$

That is to say $(j\ m\ j\ m \mid J\ M) = 0$, if $2j - J$ is odd.

We demand that state functions such as $\Phi_{JM}(j_a\ ,\, j_b)$ and $\Phi_{J'M'}(j_a\ ,\, j_b)$ are normalised and orthogonal i. e.

$$\int \Phi_{JM}{}^*(j_a\,,j_b)\;\Phi_{J'M'}(j_a\,,j_b)\;d\sigma = \delta_{JJ'}\,\delta_{MM'}. \qquad \text{(A1.9)}$$

In (A1.9) the integration comprises all variables of the functions φ_{jm}. By inserting the relation (A1.1) and by means of the given orthonormality one obtains

$$\sum_{m_a\,m_b} (j_a\;m_a\;j_b\;m_b \mid J\;M)\;(j_a\;m_a\;j_b\;m_b \mid J'\;M') = \delta_{JJ'}\,\delta_{MM'}. \qquad \text{(A1.10)}$$

Making use of (A1.4) up to (A1.6) one obtains

$$\sum_{m_b\,M} (j_a\;m_a\;j_b\;m_b \mid J\;M)\;(j_a'\;m_a'\;j_b\;m_b \mid J\;M) = (2J+1)(2j_a+1)^{-1}\,\delta_{j_a j_a'}\,\delta_{m_a m_a'}. \qquad \text{(A1.11)}$$

Multiplying both sides of (A1.1) by $\varphi_{j_a m_a'}{}^*\varphi_{j_b m_b'}{}^*$ and integrating yield the relation

$$\int \varphi_{j_a m_a'}{}^*\varphi_{j_b m_b'}{}^*\;\Phi_{JM}(j_a\,,j_b)\;d\sigma = (j_a\;m_a'\;j_b\;m_b' \mid J\;M). \qquad \text{(A1.12)}$$

Because of the linearity of (A1.1) one expects an inverse transformation with real coefficients. We write it with conjugate complex functions like this

$$\varphi_{j_a m_a}{}^*\varphi_{j_b m_b}{}^* = \sum_{J'M'} C(j_a\;m_a\;j_b\;m_b\;J'\;M')\;\Phi_{J'M'}{}^*(j_a\,,j_b).$$

Multiplying by $\Phi_{JM}(j_a\,,j_b)$ and integrating one obtains with the aid of (A1.12)

$$C(j_a\;m_a\;j_b\;m_b\;J\;M) = \int \varphi_{j_a m_a}{}^*\varphi_{j_b m_b}{}^*\;\Phi_{JM}(j_a\,,j_b)\;d\sigma = (j_a\;m_a\;j_b\;m_b \mid J\;M). \qquad \text{(A1.13)}$$

Thus, we have

$$\varphi_{j_a m_a}\varphi_{j_b m_b} = \sum_{JM} (j_a\;m_a\;j_b\;m_b \mid J\;M)\;\Phi_{JM}(j_a\,,j_b) \qquad \text{(A1.14)}$$

as the inverse transformation to (A1.1). Multiplying (A1.14) by the expression $\varphi_{j_a m_a'}{}^*\varphi_{j_b m_b'}{}^* = \sum_{J'M'} (j_a\;m_a'\;j_b\;m_b' \mid J'\;M')\;\Phi_{J'M'}{}^*(j_a\,,j_b)$ and integrating one obtains the relation

$$\sum_{JM} (j_a\;m_a\;j_b\;m_b \mid J\;M)\;(j_a\;m_a'\;j_b\;m_b' \mid J\;M) = \delta_{m_a\,m_a'}\,\delta_{m_b\,m_b'} \qquad \text{(A1.15)}$$

in analogy with (A1.10). We give the explicit expressions of frequently employed Clebsch-Gordan coefficients

$$(j_a\;m_a\;j_b\;m_b \mid 0\;0) = (-1)^{j_a-m_a}(2j_a+1)^{-1/2}\,\delta_{j_a j_b}\,\delta_{m_a,\,-m_b}\,,$$

$$(2\;m\;2,\,-m \mid 1\;0) = (-1)^{-m}m/\surd 10, \qquad \text{(A1.16)}$$

$$(2,\,m-1,\,2,\,-m \mid 1,\,-1) = (-1)^{m}\,\surd(2+m)\surd(3-m)/\surd 20,$$

$$(2\;0\;2\;m \mid 2\;m) = (m^2-2)/\surd 14,$$

$$(j_a,\,m_a = j_a,\,j_b,\,m_b = j_b \mid J = j_a + j_b,\,M = j_a + j_b) = +1. \qquad \text{(A1.17)}$$

The last coefficient shows that the product of two states $|j,\;m = j\,\rangle$ is the stretched joint state lying in the z-direction. The choice of the positive sign on the right hand side of (A1.17) is arbitrary.

The symmetry properties of the Clebsch-Gordan coefficients are made visible especially by means of the 3-j symbol $\begin{pmatrix} j_a & j_b & J \\ m_a & m_b & -M \end{pmatrix}$, which is defined as follows

$$(j_a\, m_a\, j_b\, m_b \mid J\, M) = (-1)^{j_a - j_b + M} \sqrt{(2J + 1)}\, (^{j_a}{}_{m_a}\, ^{j_b}{}_{m_b}\, ^{J}{}_{-M}). \qquad (A1.18)$$

The value of the 3-*j* symbol does not alter when its columns – here they are arranged at an angle – undergo en even number of interchanges. If this number is odd or if all projections m_a, m_b and $-M$ change their signs simultaneously, the new 3-*j* symbol differs from the original one in the phase factor $(-1)^{j_a + j_b - J}$. Edmonds (1964) and (1996) has given algebraic expressions for some 3-*j* symbols. Numerical tables for 3-*j* symbols were published by Rotenberg, Bivins et al. (1959) and Brussaard and Glaudemans (1977).

We come back to the statement (A1.1) and go into the angular momentum operators. If the spin operators $\boldsymbol{J_{za}} \equiv \boldsymbol{J_{0a}}$ and $\boldsymbol{J_{zb}} \equiv \boldsymbol{J_{0b}}$ act on the states $\varphi_{j_a m_a}$ and $\varphi_{j_b m_b}$ according to

$$\boldsymbol{J_{0a}}\, \varphi_{j_a m_a} = m_a\, \varphi_{j_a m_a},\ \boldsymbol{J_{0b}}\, \varphi_{j_b m_b} = m_b\, \varphi_{j_b m_b} \text{ and} \qquad (A1.19)$$

$$\boldsymbol{J_{0a}} \varphi_{j_b m_b} = \varphi_{j_b m_b}\, \boldsymbol{J_{0a}}\ ,\ \boldsymbol{J_{0b}} \varphi_{j_a m_a} = \varphi_{j_a m_a}\, \boldsymbol{J_{0b}} \qquad (A1.20)$$

the sum operator

$$\boldsymbol{J_0} = \boldsymbol{J_{0a}} + \boldsymbol{J_{0b}} \qquad (A1.21)$$

satisfies the following eigenvalue equation with the joint expression (A1.1) or (A1.3)

$$\boldsymbol{J_0} \Phi_{JM}(j_a\ , j_b) \equiv \boldsymbol{J_0}\, [\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_M = M\, [\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_M$$
(A1.22)

which can be seen with the help of (A1.1), (A1.19) and (A1.20). The other “ladder“ components of $\underline{\boldsymbol{J}}$ read according to (8.6) as follows $\boldsymbol{J_1} = -(\boldsymbol{J_x} + i\boldsymbol{J_y})/\sqrt{2}$ and $\boldsymbol{J_{-1}} = (\boldsymbol{J_x} - i\boldsymbol{J_y})/\sqrt{2}$. They meet the condition (13.26)

$$\boldsymbol{J_{\pm 1,a}}\, \varphi_{j_a m_a} = -(\pm)((j_a(j_a + 1) - m_a(m_a \pm 1))/2)^{1/2}\, \varphi_{j_a,\, m_a \pm 1}\ ,$$

which holds analogously for $\boldsymbol{J_{\pm 1,b}}\, \varphi_{j_b m_b}$. In analogy with (A1.21) we make up the operator

$$\boldsymbol{J_{\pm 1}} = \boldsymbol{J_{\pm 1,a}} + \boldsymbol{J_{\pm 1,b}} \qquad (A1.23)$$

and expect that the similar relation

$$\boldsymbol{J_{\pm 1}}[\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_M =$$

$$-(\pm)((J(J + 1) - M(M \pm 1))/2)^{1/2}[\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_{M \pm 1} \qquad (A1.24)$$

will hold. We write it in detail as follows

$$\sum_{m_a\, m_b} (j_a\, m_a\, j_b\, m_b \mid J\, M)\, \{(j_a(j_a + 1) - m_a(m_a \pm 1))^{1/2} \varphi_{j_a,\, m_a \pm 1} \varphi_{j_b m_b} +$$

$$(j_b(j_b + 1) - m_b(m_b \pm 1))^{1/2} \varphi_{j_a m_a} \varphi_{j_b m_b \pm 1}\, \} = \qquad (A1.25)$$

$$(J(J + 1) - M(M \pm 1))^{1/2}[\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_{M \pm 1}\ .$$

The relation (A1.25) cannot be explained solely by the properties of the Clebsch-Gordan coefficients put together in this appendix. In contrary, they constitute a special condition for these coefficients. Making use of the orthonormality of the state functions, from (A1.25) one obtains recursion formulas for the Clebsch-Gordan coefficients. They serve to calculate them numerically.

In case the operator of the total spin, $\boldsymbol{J}^2$ (8.8), acts on a coupled state, from (A1.22), (A1.24) and (8.8) follows the well-known relation

$$\boldsymbol{J}^2\,[\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_M = (\boldsymbol{J_0}^2 - \boldsymbol{J_0} - 2\boldsymbol{J_1}\boldsymbol{J_{-1}})\,[\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_M =$$

$$J(J+1)\,[\varphi_{j_a m_a} \times \varphi_{j_b m_b}]^{(J)}{}_M\,. \qquad \text{(A1.26)}$$

A2 Symmetry properties of coupled spin states of identical objects

In this section the selection rules for the total spin of a coupled state of identical objects - fermions or bosons - will be looked into. According to (A1.1) such a coupling reads as follows

$$\Phi_{JM}(j\,j,1\;2) = \sum_{m\,m'} (j\;m\;j\;m' \mid J\;M)\;\varphi_{jm}(1)\varphi_{jm'}(2). \qquad \text{(A2.1)}$$

In (A2.1) provisionally we are considering these objects as distinguishable and label them with numbers. We now interchange both objects - that is to say all their variables - and obtain the following joint state

$$\Phi_{JM}(j\,j,2\;1) = \sum_{m\,m'} (j\;m\;j\;m' \mid J\;M)\;\varphi_{jm}(2)\varphi_{jm'}(1).$$

With (A1.7) we have

$$\Phi_{JM}(j\,j,2\;1) = (-1)^{2j-J} \sum_{m\,m'} (j\;m'\;j\;m \mid J\;M)\;\varphi_{jm'}(1)\varphi_{jm}(2) =$$

$$(-1)^{2j-J}\,\Phi_{JM}(j\,j,1\;2). \qquad \text{(A2.2)}$$

First, we apply (A2.2) to fermions - for example to nucleons. They obey the Pauli principle i. e. the function of the whole state changes its sign if both fermions are interchanged as follows

$$\Phi_{JM}(j\,j,2\;1) = -\Phi_{JM}(j\,j,12) \;\text{ i. e.} \qquad \text{(A2.3)}$$

$$(-1)^{2j-J} = -1. \qquad \text{(A2.4)}$$

Fermions have half-integer spin j, that is why $2j$ is odd. Consequently, the equations (A2.3) and (A2.4) are satisfied if J is even. That is to say, a pair of identical fermions can be represented by (A2.1) if J is even.

On the other hand, bosons behave symmetrically with respect to interchanging i. e. for a coupled pair of identical bosons the following relation holds

$$\Phi_{JM}(j\,j,2\;1) = +\Phi_{JM}(j\,j,12) \;\text{ or} \qquad \text{(A2.5)}$$

$$(-1)^{2j-J} = +1. \qquad \text{(A2.6)}$$

Because here j is integer the total spin (or angular momentum) J must also be even. Specially for j = 2 the quantity J can amount to

$$J = 0, 2 \text{ or } 4. \qquad (A2.7)$$

A3 Racah-coefficients and 6-*j* symbols

Three state functions with spins can be coupled by combining the first and the second function and by adding the third one to the joint function. An other way to do it consists in coupling the second and the third function and in adding the result to the first one. By means of the Racah coefficients, it is possible to represent one configuration by linear combinations of the other type.

Three eigenfunctions of the spin - or angular momentum - operators $\boldsymbol{J}^2$ and $\boldsymbol{J_z}$ are given and coupled in the following way (A1.1, A1.3)

$$[[\varphi_{j_a} \times \varphi_{j_b}]^{(J_{ab})} \times \varphi_{j_c}]^{(J)}{}_M = \qquad (A3.1)$$

$$\sum_{m_a\, m_b\, M_{ab}\, m_c} (j_a\, m_a\, j_b\, m_b \mid J_{ab}\, M_{ab})\, (J_{ab}\, M_{ab}\, j_c\, m_c \mid J\, M)\; \varphi_{j_a\, m_a}\, \varphi_{j_b\, m_b}\, \varphi_{j_c\, m_c}\; .$$

With the help of (A1.14), we recouple the triple of functions on the right hand side in (A3.1) this way

$$[[\varphi_{j_a} \times \varphi_{j_b}]^{(J_{ab})} \times \varphi_{j_c}]^{(J)}{}_M =$$

$$\sum_{m_a\, m_b\, M_{ab}\, m_c} (j_a\, m_a\, j_b\, m_b \mid J_{ab}\, M_{ab})\, (J_{ab}\, M_{ab}\, j_c\, m_c \mid J\, M) \cdot \qquad (A3.2)$$

$$\sum_{J_{bc}\, M_{bc}} (j_b\, m_b\, j_c\, m_c \mid J_{bc}\, M_{bc}) \cdot$$

$$\sum_{J'\, M'} (j_a\, m_a\, J_{bc}\, M_{bc} \mid J'\, M')\, [\varphi_{j_a} \times [\varphi_{j_b} \times \varphi_{j_c}]^{(J_{bc})}]^{(J')}{}_{M'}\, \delta_{JJ'}\, \delta_{MM'}\; .$$

The symbol $\delta_{JJ'} \cdot \delta_{MM'}$ takes into account that both sides of (A3.2) are eigenfunctions of $\boldsymbol{J}^2$ and $\boldsymbol{J_z}$ with the same corresponding eigenvalues. We can write (A3.2) like this

$$[[\varphi_{j_a} \times \varphi_{j_b}]^{(J_{ab})} \times \varphi_{j_c}]^{(J)}{}_M = \sum_{J_{bc}} U(j_a\, j_b\, J\, j_c,\, J_{ab}\, J_{bc})\, [\varphi_{j_a} \times [\varphi_{j_b} \times \varphi_{j_c}]^{(J_{bc})}]^{(J)}{}_M \qquad (A3.3)$$

with $\quad U(j_a\, j_b\, J\, j_c;\, J_{ab}\, J_{bc}) =$

$$\sum_{m_a\, m_b} (j_a\, m_a\, j_b\, m_b \mid J_{ab},\, m_a + m_b)\, (J_{ab},\, m_a + m_b,\, j_c,\, M - m_a - m_b \mid J\, M) \cdot$$

$$(j_b\, m_b\, j_c,\, M - m_a - m_b \mid J_{bc},\, M - m_a)\, (j_a\, m_a\, J_{bc},\, M - m_a \mid J\, M) \equiv \qquad (A3.4)$$

$$\sqrt{(2J_{ab} + 1)}\, \sqrt{(2J_{bc} + 1)}\; W(j_a\, j_b\, J\, j_c\, ;\, J_{ab}\, J_{bc}).$$

The equation (A3.3) reveals that the coupling of three spin eigenfunctions according to (A3.1) can be represented as a linear combination of couplings where the second and the third functions are combined first. The factors U (A3.4) are named normalised Racah-coefficients. Apart from two factors they agree with the common Racah-coefficients W (A3.4). The following inverse relation can be shown

$$[\varphi_{j_a} \times [\varphi_{j_b} \times \varphi_{j_c}]^{(J_{bc})}]^{(J)}{}_M = \sum_{J_{ab}} U(j_a\, j_b\, J\, j_c,\, J_{ab}\, J_{bc})\, [[\varphi_{j_a} \times \varphi_{j_b}]^{(J_{ab})} \times \varphi_{j_c}]^{(J)}{}_M. \qquad \text{(A3.5)}$$

In order to visualise the symmetry properties of the Racah-coefficients, the 6-*j* symbol has been created as follows

$$\begin{Bmatrix} j_a & j_b & j_c \\ l_a & l_b & l_c \end{Bmatrix} = (-1)^{-j_a - j_b - l_a - l_b} (2j_c + 1)^{-1/2}(2l_c + 1)^{-1/2}\, U(j_a\, j_b\, l_b\, l_a\, ;\, j_c\, l_c) \equiv$$

$$(-1)^{-j_a - j_b - l_a - l_b}\, W(j_a\, j_b\, l_b\, l_a\, ;\, j_c\, l_c). \qquad \text{(A3.6)}$$

The value of the 6-*j* symbol is unaltered when the columns - here they are arranged at an angle - are interchanged. It remains also the same when two columns are turned upside down. The following triples of quantities in the 6-*j* symbol have to form a triangle

$(j_a\, j_b\, j_c)$, $(j_a\, l_b\, l_c)$, $(l_a\, l_b\, j_c)$ and $(l_a\, j_b\, l_c)$

which is indicated by lines in the symbols as follows

$$\begin{Bmatrix} j_a & \text{---} & j_b & \text{----} & j_c \\ l_a & & l_b & & l_c \end{Bmatrix}, \begin{Bmatrix} j_a & & j_b & & j_c \\ l_a & \backslash & l_b & \text{---} & l_c \end{Bmatrix}, \begin{Bmatrix} j_a & & j_b & & j_c \\ l_a & \text{---} & l_b & / & l_c \end{Bmatrix}, \begin{Bmatrix} j_a & & j_b & & j_c \\ l_a & / & l_b & \backslash & l_c \end{Bmatrix}. \qquad \text{(A3.7)}$$

From the orthonormality of the expression (A3.3), follows this orthogonality relation

$$\sum_{j_c} (2j_c + 1) \begin{Bmatrix} j_a & j_b & j_c \\ l_a & l_b & l_c \end{Bmatrix} \begin{Bmatrix} j_a & j_b & j_c \\ l_a & l_b & l_{c'} \end{Bmatrix} = \delta_{l_c\, l_{c'}} / (2l_c + 1). \qquad \text{(A3.8)}$$

A similar relation reads like this

$$\sum_{l_c} (-1)^{l_c} (2l_c + 1) \cdot \begin{Bmatrix} j_a & j & j_c \\ j & l_b & l_c \end{Bmatrix}^2 = \begin{Bmatrix} j_a & j & j_c \\ l_b & j & j_c \end{Bmatrix} \text{ for } j_c\, . \qquad \text{(A3.9)}$$

With the orthonormality of the Clebsch-Gordan coefficients the equations (A3.4) and (A3.6) can be brought in the following form

$$(-1)^{-l_a + l_b + n_c} \sqrt{(2l_a + 1)}\sqrt{(2l_b + 1)}\, (l_a\, n_a\, l_b\, n_b \mid l_c\, n_c) \begin{Bmatrix} l_a & l_b & l_c \\ j_a & j_b & j_c \end{Bmatrix} =$$

$$\sum_{m_a\, m_b\, m_c} (-1)^{j_a + j_b + j_c + m_a + m_b + m_c}\, . \qquad \text{(A3.10)}$$

$$(j_a\, m_a\, j_b, -m_b \mid l_c\, , -n_c)\, (j_b\, m_b\, j_c, -m_c \mid l_a, -n_a)\, (j_c\, m_c\, j_a, -m_a \mid l_b, -n_b).$$

Furthermore the relation

$$\sum_j (2j + 1) \begin{Bmatrix} j_1 & j_2 & j' \\ j_1 & j_2 & j \end{Bmatrix} = (-1)^{2(j_1 + j_2)}. \qquad \text{(A3.11)}$$

holds. Some 6-*j* symbols are simple algebraic expressions. If l_c is zero, according to the triangle conditions (A3.7) we have $l_b = j_a$ and $l_a = j_b$. Then the 6-*j* symbol has the form

$$\begin{Bmatrix} j_a & j_b & j_c \\ j_b & j_a & 0 \end{Bmatrix} = (-1)^{j_a + j_b + j_c} (2j_a + 1)^{-1} (2j_c + 1)^{-1}. \qquad \text{(A3.12)}$$

The zero can be brought to any position by using the interchanging rules mentioned above. We refer to a further special 6-*j* symbol (Edmonds 1964, p. 154)

$$\begin{Bmatrix} j_a & j_a & 1 \\ j_b & j_b & j_c \end{Bmatrix} = (-1)^{j_a+j_b+j_c+1}[j_a(j_a + 1) + j_b(j_b + 1) - j_c(j_c + 1)]\cdot$$

$$[4j_a(j_a + 1)(2j_a + 1)j_b(j_b + 1)(2j_b + 1)]^{-1/2}. \qquad \text{(A3.13)}$$

In the following table, numerical expressions of some 6-*j* symbols are put together (Rotenberg, Bivins et al. 1959), which appear in the IBM

J	J'	$\begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J' \end{Bmatrix}$
2	0	1/5
2	1	-1/10
2	2	-3/70
2	3	4/35
2	4	2/35
4	0	1/5
4	1	2/15
4	2	2/35
4	3	1/70
4	4	1/630.

(A3.14)

A4 The 9-*j* symbol

If four spin eigenfunctions are coupled in pairs which are coupled additionally like this

$$[[\varphi_{j_a} \times \varphi_{j_b}]]^{(J_{ab})} \times [\varphi_j \times \varphi_{j'}]^{(J')}]^{(J)}{}_M ,$$

this configuration can be represented by means of equivalent expressions where φ_{j_b} and φ_j are interchanged. After several steps one obtains the following result

$$[[\varphi_{j_a} \times \varphi_{j_b}]^{(J_{ab})} \times [\varphi_j \times \varphi_{j'}]^{(J')}]^{(J)}{}_M =$$

$$\sum_{J_a\, J_b} [(2J_{ab} + 1)(2J' + 1)(2J_a + 1)(2J_b + 1)]^{1/2} \cdot \qquad \text{(A4.1)}$$

$$\begin{Bmatrix} j_a & j_b & J_{ab} \\ j & j' & J' \\ J_a & J_b & J \end{Bmatrix} [[\varphi_{j_a} \times \varphi_j]^{(J_a)} \times [\varphi_{j_b} \times \varphi_{j'}]^{(J_b)}]^{(J)}{}_M.$$

The matrix in braces is the 9-*j* symbol. It shows certain symmetries. An even permutation of its columns - here they are arranged at an angle - or of its rows leaves its value unaffected. An odd permutation of rows or columns introduces the phase factor $(-1)^s$, where s is equal to the sum of all arguments in the 9-*j* symbol. In every column and row, the triangle condition is satisfied.

If the 9-*j* symbol contains the argument $J = 0$, the relations $J_{ab} = J'$ and $J_a = J_b \equiv J''$ hold. In this case the 6-*j* symbol reads

$$\begin{Bmatrix} j_a & j_b & J' \\ j & j' & J' \\ J'' & J'' & 0 \end{Bmatrix} = (-1)^{j_b+j+J'+J''}[(2J' + 1)(2J'' + 1)]^{-1/2}\begin{Bmatrix} j_a & j & J'' \\ j' & j_b & J' \end{Bmatrix}. \qquad \text{(A4.2)}$$

Now we replace the functions φ in (A4.1) by operators $\boldsymbol{b}$ and write the last brackets in detail with the help of (A1.1) and (A1.3). We leave the operators $\boldsymbol{b}$ in the original sequence as follows

$$[[\boldsymbol{b}_{j_a} \times \boldsymbol{b}_{j_b}]]^{(J_{ab})} \times [\boldsymbol{b}_j \times \boldsymbol{b}_{j'}]^{(J')}]^{(J)}{}_M =$$

$$\sum_{J_a J_b} [(2J_{ab} + 1)(2J' + 1)(2J_a + 1)(2J_b + 1)]^{1/2} \cdot \begin{Bmatrix} j_a & j_b & J_{ab} \\ j & j' & J' \\ J_a & J_b & J \end{Bmatrix} \cdot \qquad \text{(A4.3)}$$

$$\sum_{m_a m_b m m' M_a M_b} (j_a\, m_a\, j\, m \mid J_a\, M_a)\, (j_b\, m_b\, j'\, m' \mid J_b\, M_b)\, (J_a\, M_a\, J_b\, M_b \mid J\, M) \cdot$$

$$\boldsymbol{b}_{j_a m_a} \boldsymbol{b}_{j_b m_b} \boldsymbol{b}_{jm} \boldsymbol{b}_{j'm'}.$$

If the operators $\boldsymbol{b}$ commute, the equation (A4.3) can be transformed to the form (A4.1), otherwise this conversion is not possible.

A5 The Wigner-Eckart theorem

The IBM deals with matrix elements of coupled ***s***- and/or ***d***-operators (chapter 11 and 12). The Wigner-Eckart theorem makes possible to split up these matrix elements in a Clebsch-Gordan coefficient and an element which is independent on projection quantum numbers of the state functions and of the operators.

We start from the properties of the spherical harmonics $Y_{lm}(\underline{\rho})$ depending on the direction vector $\underline{\rho} = \underline{r}/r$. They are eigenfunctions of the angular momentum operators $\boldsymbol{J}^2$ and $\boldsymbol{J}_z$ and therefore they represent a part of the wave functions of particles in a central field. They are orthonormalised i. e.

$$\int Y^*_{lm}(\underline{\rho}) Y_{l'm'}(\underline{\rho})\, d\underline{\rho} \equiv \int Y^*_{lm}(\vartheta,\varphi) Y_{l'm'}(\vartheta,\varphi) \sin\vartheta\, d\vartheta\, d\varphi = \delta_{ll'}\, \delta_{mm'}. \qquad \text{(A5.1)}$$

Spherical harmonics have the following properties

$$Y_{lm}(\underline{\rho}) = (-1)^m\, Y^*_{l,-m}(\underline{\rho}) \qquad \text{(A5.2)}$$

and $$Y_{l_a m_a}(\underline{\rho})\, Y_{l_b m_b}(\underline{\rho}) = \sum_{LM} [(2l_a + 1)(2l_b + 1)]^{1/2}\, [4\pi(2L + 1)]^{-1/2} \cdot$$

$$(l_a\, m_a\, l_b\, m_b \mid L, -M)\, (l_a\, 0\, l_b\, 0 \mid L\, 0)\, (-1)^{-M}\, Y^*_{LM}(\underline{\rho}). \qquad \text{(A5.3)}$$

Now we make up the following integral

$$\int Y^*_{l_a m_a}(\underline{\rho}) Y_{LM}(\underline{\rho}) Y_{l_b m_b}(\underline{\rho}) d\underline{\rho} = (-1)^{m_a} \int Y_{LM}(\underline{\rho})\, Y_{l_a,-m_a}(\underline{\rho})\, Y_{l_b m_b}(\underline{\rho}) d\underline{\rho} =$$

$$\sum_{L'M'} [(2l_a + 1)(2l_b + 1)]^{1/2}\, [4\pi(2L' + 1)]^{-1/2} \cdot \qquad \text{(A5.4)}$$

$$(l_a, -m_a\, l_b\, m_b \mid L', -M')\, (l_a\, 0\, l_b\, 0 \mid L'\, 0)\, (-1)^{-M'+m_a} \int Y^*_{L'M'}(\underline{\rho}) Y_{LM}(\underline{\rho}) d\underline{\rho} =$$

$$(-1)^{-M+m_a} (l_a, -m_a\, l_b\, m_b \mid L, -M)\, [(2l_a + 1)(2l_b + 1)]^{1/2} [4\pi(2L + 1)]^{-1/2} \cdot$$

$$(l_a\, 0\, l_b\, 0 \mid L\, 0).$$

Doing this, (A5.1) and (A5.3) have been used. Making use of (A1.6) we obtain

$$\int Y^*_{l_a m_a}(\varrho) Y_{LM}(\varrho) Y_{l_b m_b}(\varrho) d\varrho = \tag{A5.5}$$

$$(-1)^{2L}(l_b\ m_b\ L\ M \mid l_a\ m_a)\ (2l_a + 1)^{-1/2}(-1)^{l_a+2l_b+L}[(2l_a + 1)(2l_b + 1)/4\pi]^{1/2}(l_a\ 0\ l_b\ 0 \mid L\ 0).$$

We write (A5.5) in the following form

$$\int Y^*_{l_a m_a}(\varrho) Y_{LM}(\varrho) Y_{l_b m_b}(\varrho) d\varrho \equiv \tag{A5.6}$$

$$\langle\ Y_{l_a m_a} \mid Y_{LM} \mid Y_{l_b m_b}\ \rangle = (-1)^{2L}(l_b\ m_b\ L\ M \mid l_a\ m_a)\ (2l_a + 1)^{-1/2}\ \langle\ Y_{l_a} \| Y_L \| Y_{l_b}\ \rangle.$$

The last quantity is named reduced matrix element. It is independent on the projection quantum numbers m_a , M and m_b and it reads

$$\langle\ Y_{l_a} \| Y_L \| Y_{l_b}\ \rangle = (-1)^{l_a+2l_b+L}[(2l_a + 1)(2l_b + 1)/4\pi]^{1/2}(l_a\ 0\ l_b\ 0 \mid L\ 0). \tag{A5.7}$$

The equations (A5.6) can be generalised. One can replace $Y_{l_a m_a}$ and $Y_{l_b m_b}$ by the eigenfunctions $\mid l_a\ m_a\ \rangle$ and $\mid l_b\ m_b\ \rangle$ of $\boldsymbol{J}^2$ and $\boldsymbol{J_z}$ with a common potential. Instead of the function Y_{LM}, which is placed in the middle of the original matrix element (A5.5), one can put a tensor operator $\boldsymbol{T}^{(L)}{}_M$ of the rank L. These operators and spherical harmonics behave equally when the co-ordinate system rotates. After a rotation the new spherical harmonics are linear combinations of spherical harmonics belonging to the old system with the same rank L. For a given rotation, the corresponding coefficients for spherical harmonics and tensor operators are identical.

As an equivalent statement, one can say that tensor operators can be coupled to an operator with a defined angular momentum with the same procedure as the spherical harmonics i. e. by making use of Clebsch-Gordan coefficients. Examples for tensor operators are the angular momentum operator $\boldsymbol{J_i}$, the nabla operator ∇, creation operators for bosons $\boldsymbol{d}^+$, $\boldsymbol{s}^+$ (see the comment on (6.9)) and the corresponding modified annihilation operators $\boldsymbol{d}^{\sim}$ and $\boldsymbol{s}$ (see the comment on (6.19) and (6.20)).

One can show that (A5.6) remains valid if all three functions are replaced in the way just mentioned like this

$$\langle\ l_a\ m_a \mid \boldsymbol{T}^{(L)}{}_{\mathrm{M}} \mid l_b\ m_b\ \rangle = (-1)^{2L}(l_b\ m_b\ L\ M \mid l_a\ m_a)\ (2l_a + 1)^{-1/2}\ \langle\ l_a \| \boldsymbol{T}^{(L)} \| l_b\ \rangle. \tag{A5.8}$$

This is the Wigner-Eckart theorem. It enables to split the matrix elements of a tensor operator in a reduced matrix element, which does not depend on the projection quantum numbers m and in a Clebsch-Gordan coefficient. It offers an essential simplification because as soon as a matrix element with a set of m-values is known it can be given for other sets of this kind. One obtains the reduced matrix element (on the right hand side of (A5.8)) by calculating the left side for one set of m-values and by remodelling.

Reduced matrix elements are also necessary for constituting the so-called coefficients of fractional parentage.

A6 A further multipole representation of the Hamiltonian

The multipole representation of the Hamilton operator according to (9.37) stands out due to an especially consequent choice of operators which permit to constitute the three special cases of the IBM in a direct manner. Frequently, in the literature the multipole representation appears, which was formulated originally by Iachello and Arima. In their review article, these authors use it again in 1987. This representation was built in the computer code PHINT (appendix 7) by O. Scholten in a simplified form.

This multipole representation contains the operators $\boldsymbol{n_d}$ and $\boldsymbol{J}^2$ (9.1), which are already present in the Hamiltonian in (9.37). Instead of $\boldsymbol{Q}^2$ (9.4) the more general form $\boldsymbol{Q}^2(\chi)$ (12.19, 15.17) is employed. Its components read

$$\boldsymbol{Q}_\mu(\chi) = \boldsymbol{d}^+_\mu \boldsymbol{s} + \boldsymbol{s}^+ \tilde{\boldsymbol{d}}_\mu + \chi\, [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}_\mu\,. \tag{A6.1}$$

Furthermore the operators $[[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(J)} \times [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(J)}]^{(0)}$ with J = 3, 4 occur, which come from the transformation of the term $[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times (\tilde{\boldsymbol{d}} \times \tilde{\boldsymbol{d}}]^{(J)}]^{(0)}$ in (7.17).

In concrete terms, we make the claim that the Hamilton operator can be written as follows

$$\boldsymbol{H} = \varepsilon'\, \boldsymbol{n_d} + \tfrac{1}{2}\, \eta\, \boldsymbol{J}^2 + \tfrac{1}{2}\, \kappa\, \boldsymbol{Q}^2(\chi) - 5\sqrt{7}\, \omega\, [[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)} \times (\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(3)}]^{(0)} +$$
$$5{\cdot}3\, \xi\, [[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(4)} \times (\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(4)}]^{(0)}. \tag{A6.2}$$

Now we are working on the individual terms in (A6.2) in order to transform this equation in the basic form (7.17). With the help of (9.28) and (9.34) we can write $\boldsymbol{Q}^2(\chi)$ in the following form

$$\boldsymbol{Q}^2(\chi)/\sqrt{5} = [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}\boldsymbol{ss} + \boldsymbol{s}^+\boldsymbol{s}^+[\tilde{\boldsymbol{d}} \times \tilde{\boldsymbol{d}}]^{(0)} +$$
$$\boldsymbol{ss}^+[d^+ \times \tilde{\boldsymbol{d}}]^{(0)} + [\tilde{\boldsymbol{d}} \times \boldsymbol{d}^+]^{(0)}\boldsymbol{s}^+\boldsymbol{s} + \chi^2 [[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)} \times [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(2)}]^{(0)} +$$
$$2\, \chi\, ([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times \tilde{\boldsymbol{d}}\boldsymbol{s}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{d}^+ \times [\tilde{\boldsymbol{d}} \times \tilde{\boldsymbol{d}}]^{(2)}]^{(0)}). \tag{A6.3}$$

For the time being we are dealing with the second line in (A6.3). According to the commutation rules for d-operators we have

$$[\tilde{\boldsymbol{d}} \times \boldsymbol{d}^+]^{(0)} = 5^{-1/2}\sum_{m=-1}^{+2} \tilde{\boldsymbol{d}}_m \boldsymbol{d}^+_{-m}(-1)^{-m} = 5^{-1/2}\sum_m \boldsymbol{d}_{-m}\boldsymbol{d}^+_{-m} =$$
$$5/\sqrt{5} + 5^{-1/2}\sum_m \boldsymbol{d}^+_m \boldsymbol{d}_m = \sqrt{5} + [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(0)}. \tag{A6.4}$$

Because of $\boldsymbol{ss}^+ = 1 + \boldsymbol{s}^+\boldsymbol{s}$ this line results in

$$\boldsymbol{ss}^+[d^+ \times \tilde{\boldsymbol{d}}]^{(0)} + [\tilde{\boldsymbol{d}} \times \boldsymbol{d}^+]^{(0)}\boldsymbol{s}^+\boldsymbol{s} =$$
$$[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(0)} + \boldsymbol{s}^+\boldsymbol{s}[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(0)} + [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(0)}\, \boldsymbol{s}^+\boldsymbol{s} + \sqrt{5}\, \boldsymbol{s}^+\boldsymbol{s} =$$
$$\boldsymbol{n_d}/\sqrt{5} + \sqrt{5}(\boldsymbol{N} - \boldsymbol{n_d}) + 2[\boldsymbol{d}^+\boldsymbol{s}^+ \times \tilde{\boldsymbol{d}}\boldsymbol{s}]^{(0)}. \tag{A6.5}$$

In order to handle the last expression in (A6.2) we need the relation (9.8)

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times (\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} =$$

$$\sum_{J'} (-1)^{J'} \sqrt{(2J+1)}\, \sqrt{(2J'+1)} \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J' \end{Bmatrix} [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)} -$$

$$(1/5)\sqrt{(2J+1)}\boldsymbol{n_d}. \qquad \text{(A6.6)}$$

We multiply both sides of (A6.6) by $\sqrt{(2J+1)}\begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J'' \end{Bmatrix}$, sum over J and obtain

$$\sum_J \sqrt{(2J+1)} \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J'' \end{Bmatrix} [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times (\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} =$$

$$\sum_{J'} (-1)^{J'} \sqrt{(2J'+1)} \left\langle \sum_J (2J+1) \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J' \end{Bmatrix} \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J'' \end{Bmatrix} \right\rangle [[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0)}$$

$$- \ (1/5) \left\langle \sum_J (2J+1) \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J'' \end{Bmatrix} \right\rangle \boldsymbol{n_d}. \qquad \text{(A6.7)}$$

The expressions in pointed brackets in (A6.7) read according to (A3.8) and (A3.11) like this

$$\sum_J (2J+1) \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J' \end{Bmatrix} \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J'' \end{Bmatrix} = \delta_{J''J'}/(2J''+1)$$

and $\quad \sum_J (2J+1) \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J'' \end{Bmatrix} = 1.$

Thus we obtain the inverse relation of (A6.6)

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')} \times [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}]^{(0))} =$$

$$(-1)^{J'} \sqrt{(2J'+1)} \sum_{J=0,2,4} \sqrt{(2J+1)} \begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J' \end{Bmatrix} [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(J)}]^{(0)} +$$

$$(-1)^{J'}(1/5)\sqrt{(2J'+1)}\boldsymbol{n_d}\,. \qquad \text{(A6.8)}$$

According to (6.6) the summands with J = 1, 3 vanish. We now insert the expressions (9.1) and (A6.3) in (A6.2) and make use of (A6.5) and (A6.8). By means of the 6-j symbol $\begin{Bmatrix} 2 & 2 & J \\ 2 & 2 & J' \end{Bmatrix}$ according to (A3.14) we obtain

$$\boldsymbol{H} = (\varepsilon' + 3\eta + (1/2)(\chi^2 - 4)\kappa + 7\omega + 9\xi)\boldsymbol{n_d} +$$

$$(-3\eta + (1/2)\chi^2\kappa - \ \omega + 9\xi)\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(0)}]^{(0)} +$$

$$\sqrt{5}(\ -(3/2)\eta - (3/28)\chi^2\kappa + 4\omega + (18/7)\xi)\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(2)}]^{(0)} +$$

$$3(\ 2\eta + (1/7)\chi^2\kappa + (1/2)\omega + (1/14)\xi)\, [[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(4)} \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(4)}]^{(0)} +$$

$$\sqrt{5}\chi\kappa\, ([[\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(2)} \times \boldsymbol{d}^\sim\boldsymbol{s}]^{(0)} + [\boldsymbol{s}^+\boldsymbol{d}^+ \times [\boldsymbol{d}^\sim \times \boldsymbol{d}^\sim]^{(2)}]^{(0)}) + \qquad \text{(A6.9)}$$

$$\sqrt{5}(1/2)\kappa\, ([\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(0)}\boldsymbol{ss} + \boldsymbol{s}^+\boldsymbol{s}^+[\boldsymbol{d}^\sim + \boldsymbol{d}^\sim]^{(0)}) +$$

$$\sqrt{5}\kappa\, [\boldsymbol{d}^+\boldsymbol{s}^+ \times \boldsymbol{d}^\sim\boldsymbol{s}]^{(0)}.$$

The comparison of (A6.9) with (7.17) yields

$$\varepsilon_s = 0,$$

$$\varepsilon_d - \varepsilon_s = \varepsilon' + 3\eta + (1/2)(\chi^2 - 4)\kappa + 7\omega + 9\xi,$$

$$c_0 = - 6\eta + \chi^2\kappa - 14\omega + 18\xi,$$

$$c_2 = - 3\eta - (3/14)\chi^2\kappa + 8\omega + (36/7)\xi,$$

$$c_4 = 4\eta + (2/7)\chi^2\kappa + \omega + (1/7)\xi,$$

$$v_2/\surd 2 = \surd 5\, \chi\, \kappa, \qquad \text{(A6.10)}$$

$$v_0/2 = (1/2)\surd 5\, \kappa,$$

$$u_2 = \surd 5\, \kappa,$$

$$u_0 = 0.$$

The ansatz (A6.2) comprises 6 parameters ε', η , κ , ω , ξ and χ (contained in ***Q***). The transformation (A6.10) reveals that the quantities ε_s and u_0 of the basic form (7.17) have been put equal to zero and that v_0 and u_2 are coupled. This reduction to six free parameters is permitted if one looks into the structure of individual nuclei disregarding the binding energies.

A7 The program package PHINT

The code PHINT calculates energies of nuclear states and reduced transition probabilities of the electromagnetic radiation in the interacting boson model. It was written by O. Scholten in FORTRAN, published in Computational Nuclear Physics 1 (O. Scholten, 1991, p.88) and made accessible in the diskette enclosed in that book.

The IBM calculation can be performed in the basic form of the Hamiltonian (7.17) with the program PCIBAXW. The input parameters (in capitals) correspond to the following coefficients in (7.17)

ε_s	0 (is put equal to zero)	
$\varepsilon_d - \varepsilon_s$	H BAR	
c_0 , c_2 , c_4	(C(1), C(2), C(3))	
$v_2/\surd 2$	F	(A7.1)
$v_0/2$	G	
$u_2/\surd 5$	CH2	
u_0	CH1.	

It is also possible to use the reduced form (7.31). In this case, the following assignments hold

ε	HBAR	
c_0' , c_2' , c_4'	C(1), C(2), C(3)	
$v_2/\sqrt{2}$	F	(A7.2)
$v_0/2$	G	
	CH2 = CH1 = 0.	

The calculation can be done in the multipole form of the Hamiltonian according to appendix 6, the coefficients of which correspond to the following input parameters

ε'	EPS	
η	ELL	
χ	CHQ/$\sqrt{5}$	
κ	QQ	(A7.3)
ω	OCT	
ξ	HEX.	

To the technical notes of O. Scholten (1991, p. 103) the following remarks have to be added

- The subroutine RDPAR in the packet PCIBALIB has to be corrected in order to enable it to read the input parameters (C(1), C(2), C(3)) as a triplet (for example (3F10.4) has to be replaced by (3F7.4)).

- Energies should be given in MeV.

- In order to compile the source program no sensitive debug option can be used (for example for the FORTRAN compiler of Microsoft only the command fl and not fl/4Yb must be given).

- Increasing the boson number N to 14 or 16 calls for changes in COMMON(READMAT) and in the corresponding DATA in various places of the programs. Tips in the comments of the program texts have to be followed.

A8 Commutators of operators such as $[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(J)}_M$

Commutators of the type $[[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(J')}_{M'} , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(J'')}_{M''}]$ are employed in connection with the angular momentum operators (chapter 8) and with Lie algebras in the IBM (chapter 14).

By means of (6.19) and (A1.1) we can write

$$[[\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(J')}_{M'} , [\boldsymbol{d}^+ \times \tilde{\boldsymbol{d}}]^{(J'')}_{M''}] = \tag{A8.1}$$

$$\sum_{m' n' m'' n''} (2\; m'\; 2\; n' \,|\, J'\, M')\, (2\; m''\; 2\; n'' \,|\, J''\, M'')\, (-1)^{n'+m''} [\boldsymbol{d}^+_{m'}\, \boldsymbol{d}_{-n'} \,,\, \boldsymbol{d}^+_{m''}\, \boldsymbol{d}_{-n''}].$$

Using the commutation relations (5.36) we obtain

$$[\boldsymbol{d}^+_{m'}\, \boldsymbol{d}_{-n'} \,,\, \boldsymbol{d}^+_{m''}\, \boldsymbol{d}_{-n''}] = -\boldsymbol{d}^+_{m''}\, \boldsymbol{d}_{-n'}\, \delta_{m',\, n''} + \boldsymbol{d}^+_{m'}\, \boldsymbol{d}_{-n''}\, \delta_{m'',\, -n'} \,. \tag{A8.2}$$

Thus, from (A8.1) and (6.19) the following relation results

$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}{}_{M'}, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J")}{}_{M"}] =$

$\sum_{m'\,n'\,m"\,n"} (2\ m'\ 2\ n' \mid J'\ M')\ (2\ m"\ 2\ n" \mid J"\ M")\cdot$

$((-1)^{n'}\ \boldsymbol{d}^+{}_{\boldsymbol{m'}}\ \boldsymbol{d}^\sim{}_{\boldsymbol{-n"}}\ \delta_{m", -n'} - (-1)^{n"}\ \boldsymbol{d}^+{}_{\boldsymbol{m"}}\ \boldsymbol{d}^\sim{}_{\boldsymbol{-n'}}\ \delta_{m', n"}\ ) =$

$$\sum_{m'\,n'\,m"\,n"\,k\,\kappa} (2\ m'\ 2\ n' \mid J'\ M')\ (2\ m"\ 2\ n" \mid J"\ M")\cdot \tag{A8.3}$$

$((2\ m'\ 2\ n" \mid k\ \kappa)[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)}{}_\kappa(-1)^{n'}\ \delta_{m", -n'} - (2\ m"\ 2\ n' \mid k\ \kappa)[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)}{}_\kappa(-1)^{n"}\delta_{m", -n'}).$

The last transformation was performed with the aid of (A1.14). By means of the relation (A3.10), one verifies the following result

$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}{}_{M'}, [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J")}{}_{M"}] = \sum_{k\,\kappa} [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)}{}_\kappa\cdot$

$(\sum_{n"} (2\ m'\ 2, -m" \mid J'\ M')\ (2\ m"\ 2\ n" \mid J"\ M")\ (2\ m'\ 2\ n" \mid k\ \kappa)\ (-1)^{-m"} -$

$(\sum_{n'} (2\ m'\ 2, n' \mid J'\ M')\ (2\ m"\ 2, -m' \mid J"\ M")\ (2\ m"\ 2\ n' \mid k\ \kappa)\ (-1)^{-m'}) =$

$$\sum_{k\,\kappa} [\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(k)}{}_\kappa \cdot \sqrt{(2J' + 1)}\sqrt{(2J" + 1)}\cdot \tag{A8.4}$$

$(J'\ M'\ J"\ M" \mid k\ \kappa)\ \{^{J'}{}_2\ {}^{J"}{}_2\ {}^{k}{}_2\}\ (-1)^k\ (1 - (-1)^{J'+J"+k}).$

A9 The commutator $[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}{}_{M'}, [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)}{}_M]$

This commutator is used in section 14.3. It is related to the commutator we dealt with in appendix A8. Analogously we transform as follows

$$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}{}_{M'}, [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)}{}_M] = \tag{A9.1}$$

$\sum_{m'\,n'\,m\,n} (2\ m'\ 2\ n' \mid J'\ M')\ (2\ m\ 2\ n \mid J\ M)\ (-1)^{n'}\ [\boldsymbol{d}^+{}_{\boldsymbol{m'}}\ \boldsymbol{d}_{\boldsymbol{-n'}}, \boldsymbol{d}^+{}_{\boldsymbol{m}}\ \boldsymbol{d}^+{}_{\boldsymbol{n}}]\ .$

With $$[\boldsymbol{d}^+{}_{\boldsymbol{m'}}\ \boldsymbol{d}_{\boldsymbol{-n'}}, \boldsymbol{d}^+{}_{\boldsymbol{m}}\ \boldsymbol{d}^+{}_{\boldsymbol{n}}] = \boldsymbol{d}^+{}_{\boldsymbol{m'}}\ \boldsymbol{d}^+{}_{\boldsymbol{n}}\ \delta_{-n', m} + \boldsymbol{d}^+{}_{\boldsymbol{m'}}\boldsymbol{d}^+{}_{\boldsymbol{m}}\ \delta_{-n', n} \tag{A9.2}$$

one obtains by means of (A1.4)

$[[\boldsymbol{d}^+ \times \boldsymbol{d}^\sim]^{(J')}{}_{M'}, [\boldsymbol{d}^+ \times \boldsymbol{d}^+]^{(J)}{}_M] =$

$$\sum_{m\,n} (2\ m\ 2\ n \mid J\ M) \sum_{m'} (2\ m'\ 2, -m \mid J'\ M')\ (-1)^{-m}\boldsymbol{d}^+{}_{\boldsymbol{m'}}\ \boldsymbol{d}^+{}_{\boldsymbol{n}} + \tag{A9.3}$$

$(-1)^J \sum_{m\,n} (2\ m\ 2\ n \mid J\ M) \sum_{m'} (2\ m'\ 2, -m \mid J'\ M')\ (-1)^{-n}\boldsymbol{d}^+{}_{\boldsymbol{m'}}\ \boldsymbol{d}^+{}_{\boldsymbol{m}} =$

$2 \sum_m (2\ m\ 2, M - m \mid J\ M)\ (2, M' + m, 2, -m \mid J'\ M')\ (-1)^m \boldsymbol{d}^+{}_{\boldsymbol{M'+m}}\ \boldsymbol{d}^+{}_{\boldsymbol{M-m}}\ .$

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Index